Part V

Extensions

23

Digraph analogues of the Tutte polynomial

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Synopsis

This chapter considers analogues of the Tutte polynomial for directed graphs. Although no fully satisfactory analogue of the Tutte polynomial exists for directed graphs, this chapter discusses several candidates that have been proposed.

- The cover polynomial and its multivariate generalizations—the cycle-path indicator polynomial and the path-cycle symmetric function.
- Tutte invariants of alternating dimaps.
- Various digraph polynomials of Gordon and Traldi.
- The B-polynomial of Awan and Bernardi.

23.1 Introduction

A *directed graph* or *digraph* is a graph equipped with an orientation on each edge. It is natural to ask if there is a digraph analogue of the Tutte polynomial. Several proposals have been made, as discussed in this chapter. However, none of them seems to lay claim to being *the* Tutte polynomial of a digraph, so perhaps the correct analogue remains to be discovered, or does not exist.

One key difficulty is that it is not obvious how to define the *contraction* D/e of a digraph D by an edge e. Chung and Graham [320] define contraction in a way that prevents the creation of directed paths and cycles in D/e that do not arise from directed paths and cycles in D. This allows them to define a polynomial called the *cover polynomial* that obeys a deletion–contraction relation and has interesting connections to the chromatic polynomial and classical rook theory. There are also multivariate generalizations of the cover polynomial due to D'Antona and Munarini [359] and Chow [316].

For *alternating dimaps*, which are special kinds of digraphs embedded in a surface, Farr [480] has defined three minor operations and a notion of an "extended Tutte invariant" that generalizes the Tutte polynomial of a planar graph.

Instead of focusing on the definition of contraction, one can instead seek analogues of the corank–nullity definition of the Tutte polynomial. In this direction Gordon and Traldi [574, 575] have defined several possible digraph analogues of the Tutte polynomial. More about some of these polynomials may be found in Chapter 33.

Very recently, Awan and Bernardi [57] have defined the *B-polynomial* of a digraph. If G is a graph and D is the digraph obtained by replacing each undirected edge by a bidirected edge, then the B-polynomial of D is equivalent to the Tutte polynomial of G. However, the deletion-contraction recurrence for the B-polynomial does not express it in terms of B-polynomials of digraphs with fewer edges, so there is no universality property.

Throughout this chapter, we write $V(D)$ and $E(D)$ for the vertex and edge sets of a digraph D. If the end vertices of an edge are u and v then we write $u \to v$ and think of the edge as being directed from u (the *tail*) to v (the *head*). The reader is cautioned that in different sections, digraphs may have additional restrictions placed on them.

23.2 The cover polynomial

In this section, unless otherwise stated, digraphs do not have multiple directed edges, but may have loops and may have oppositely directed edges $u \to v$ and $v \to u$.

Chung and Graham [320] defined a contraction operation on digraphs that leads, via a deletion–contraction relation, to a polynomial called the *cover polynomial*.

Definition 23.1. If e is an edge of D, then the *deletion* $D\backslash e$ is the digraph obtained by deleting e from D.

Definition 23.2. If $e = u \to v$ is an edge of D from a vertex u to a vertex v, and $u \neq v$, then the *contraction* D/e is the digraph obtained by deleting all edges leaving u (including e) and all edges entering v, and then merging u and v into a single vertex. See Figure 23.1. If $u = v$, i.e., if e is a loop, then the contraction D/e is the digraph obtained by deleting v and all its incident edges.

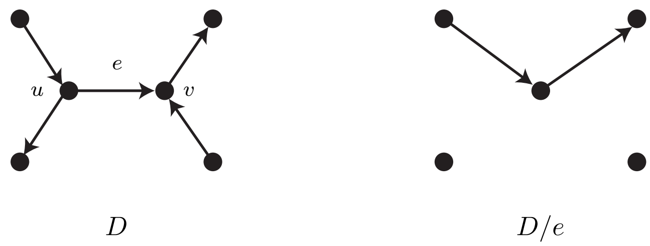

FIGURE 23.1: Chung–Graham contraction by a non-loop e.

Definition 23.3. The *cover polynomial* $C(D; x, y)$ of a digraph D is defined recursively as follows.

1. If I_n is the digraph with $n > 0$ independent vertices and no edges, then

$$C(I_n) = x^{\underline{n}} := x(x-1)\cdots(x-n+1). \tag{23.1}$$

   For $n = 0$, define $C(I_0) = 1$.

2. If e is an edge of D that is not a loop, then

$$C(D) = C(D\backslash e) + C(D/e). \tag{23.2}$$

3. If e is a loop then

$$C(D) = C(D\backslash e) + yC(D/e). \tag{23.3}$$

Chung and Graham show that $C(D; x, y)$ is well-defined, i.e., independent of the order in which edges are chosen in the above recursive procedure.

Example 23.4. Let D be a digraph on two vertices u and v, with a loop on u and two edges $u \to v$ and $v \to u$. Then $C(D; x, y) = x^2 + xy + x + y$.

Definition 23.5. A *path-cycle cover* of a digraph D is a subgraph consisting of a disjoint union of directed paths and directed cycles such that every vertex of D belongs to exactly one directed path or cycle. A path may have zero edges and a loop counts as a cycle. We write $c_D(i, j)$ for the number of path-cycle covers of D with i directed paths and j directed cycles.

The cover polynomial derives its name from the following basic result from [320].

Theorem 23.6. *For a digraph D,*

$$C(D; x, y) = \sum_{i,j} c_D(i, j) x^{\underline{i}} y^j.$$

Corollary 23.7. *If a digraph D is formed by joining the disjoint digraphs D_1 and D_2 with all possible edges $v_1 \to v_2$ with $v_1 \in V(D_1)$ and $v_2 \in V(D_2)$, then $C(D) = C(D_1)C(D_2)$.*

23.2.1 Connections with classical polynomials

If $D = (V, E)$, we may visualize $V \times V$ as a square chessboard, and $E \subseteq V \times V$ as a subset of that chessboard. We obtain a connection between the cover polynomial and *rook theory*, which is the theory of enumerating the number of ways of placing non-attacking rooks on E.

Theorem 23.8. *If $D = (V, E)$ has n vertices, then the number of $(n - i)$-element subsets of E with no two elements in the same row or column is $\sum_j c_D(i, j)$.*

The theorem above is from [320]. In the context of rook theory, Gessel [519] had earlier defined a two-variable polynomial that is very similar to the cover polynomial.

Definition 23.9. If D is a digraph with n vertices then the *cycle rook polynomial* $r(D; x, y)$ is defined by

$$r(D; x, y) := \sum_{i,j} (-1)^{n-i} c_D(i, j) x^i y^j. \tag{23.4}$$

The single-variable polynomial from rook theory most closely related to the cover polynomial is the *n-factorial (rook) polynomial* of Goldman, Joichi, and White [554, 555], which in our notation coincides with $C(D; x, 1)$. Among other things, Goldman, Joichi, and White showed that the n-factorial polynomial factors into linear factors for *increasing Ferrers boards*, (i.e., a subset of the chessboard such that all squares below or to the right of an included square are also included), and *decreasing Ferrers boards* (i.e., a subset of the chessboard such that all squares below or to the left of an included square are also included). Dworkin [429] generalized this result to the cover polynomial (the result is stated in the following theorem). For increasing Ferrers boards, the result was proved independently by Haglund (unpublished).

Theorem 23.10. *The cover polynomials of increasing Ferrers boards and decreasing Ferrers boards factor completely into linear factors.*

The n-factorial polynomial of a board determines the n-factorial polynomial of the complementary board. This fact extends to the cover polynomial, as proved independently by Gessel (unpublished) and Chow [316].

The *complement* $\overline{D}$ of a digraph D is the digraph with the same vertex set as D and with an edge $u \to v$ (or loop $u \to u$) precisely when there is *not* an edge $u \to v$ (or loop $u \to u$) in D.

Theorem 23.11. *Let D be a digraph with n vertices and $\overline{D}$ be its complement. Then $C(\overline{D}; x, y) = (-1)^n C(D; -x - y, y)$.*

As Chung and Graham noted, the cover polynomial is also related to the chromatic polynomial.

Theorem 23.12. *Let $P = (V, \prec)$ be a partially ordered set. Let $D(P)$ be the digraph with vertex set V that has an edge $u \to v$ precisely when $u \prec v$ in P. Let $G(P)$ be the* incomparability graph *of P, i.e., the undirected graph with vertex set V in which u and v are adjacent precisely when u and v are incomparable in P. Then*

$$C(D(P); x, 1) = \chi(G(P); x).$$

23.2.2 The cycle-path indicator polynomial

D'Antona and Munarini [359] have considered some digraph polynomials that are closely related to the cover polynomial.

Definition 23.13. Let D be a digraph. The *geometric cover polynomial* $\hat{C}(D; x, y)$ is defined by

$$\hat{C}(D; x, y) := \sum_{i,j} c_D(i, j) x^i y^j. \tag{23.5}$$

Warning: D'Antona and Munarini write $C(D; x, y)$ for what in our notation is $\hat{C}(D; y, x)$ and write $\tilde{C}(D; x, y)$ for what in our notation is $C(D; y, x)$.

The polynomial $\hat{C}(D)$ satisfies the deletion–contraction relations (23.2) and (23.3) (with $\hat{C}(D)$ replacing $C(D)$ in the relations). However, $\hat{C}(D)$ satisfies a different base case than $C(D)$: in place of (23.1) we have $\hat{C}(I_n) = x^n$. Also, $\hat{C}(D; x, y) = (-1)^n r(D; -x, y)$, where $r(D)$ is as defined in (23.4).

Example 23.14. For the digraph D in Example 23.4, $\hat{C}(D; x, y) = x^2 + xy + 2x + y$.

More generally, D'Antona and Munarini define a multivariate *cycle-path indicator polynomial* as follows.

Definition 23.15. A *vertex-weighted* digraph is a digraph, possibly with multiple edges, that has a nonnegative integer weight $w(v)$ associated with each vertex v.

Definition 23.16. Let D be a vertex-weighted digraph. Let $x_1, y_1, x_2, y_2, \ldots$ be independent indeterminates. If β is a directed path or a directed cycle with k vertices $v_1, \ldots, v_k$, then define

$$\mathrm{Ind}(\beta) := \begin{cases} x_{k+w(v_1)+\cdots+w(v_k)}, & \text{if } \beta \text{ is a path;} \\ y_{k+w(v_1)+\cdots+w(v_k)}, & \text{if } \beta \text{ is a cycle.} \end{cases}$$

If $\mathscr{C}$ is a path-cycle cover of D, then define

$$\mathrm{Ind}(\mathscr{C}) := \prod_{\beta \in \mathscr{C}} \mathrm{Ind}(\beta),$$

where the product is over all directed paths and cycles β in $\mathscr{C}$. Finally, define the *cycle-path indicator polynomial* of D by

$$\mathrm{Ind}(D) := \sum_{\mathscr{C}} \mathrm{Ind}(\mathscr{C}),$$

where the sum is over all path-cycle covers of D.

Example 23.17. For the D of Example 23.4, and assuming all the vertex weights are zero, $\mathrm{Ind}(D) = x_1^2 + x_1 y_1 + 2x_2 + y_2$.

It is easy to see that if all the vertex weights are zero and we set $x_i = x$ and $y_i = y$ for all i, then the cycle-path indicator polynomial coincides with the geometric cover polynomial.

The main result of [359] is a deletion–contraction relation for $\mathrm{Ind}(D)$.

Definition 23.18. Let D be a vertex-weighted digraph with an edge e. The vertex-weighted digraphs $D\backslash e$ and D/e are formed by following Definitions 23.1 and 23.2 and weighting the resulting digraphs as follows. The vertex weights of $D\backslash e$ are the same as the vertex weights of D. If u and v are the original end vertices of e, then the weight of the vertex that they merge into in forming D/e is defined to be $w(u) + w(v) + 1$.

Thus the vertex weights keep a record of edges that have been contracted.

Theorem 23.19. *Let D be a vertex-weighted digraph with an edge e. If e is not a loop, then*

$$\mathrm{Ind}(D) = \mathrm{Ind}(D\backslash e) + \mathrm{Ind}(D/e).$$

If e is a loop on the vertex v, then

$$\mathrm{Ind}(D) = \mathrm{Ind}(D\backslash e) + y_{w(v)+1}\,\mathrm{Ind}(D/e).$$

23.2.3 Computational complexity

Nederlof [875, Section 5] has given a polynomial-space algorithm for computing the cover polynomial. In the other direction, Bläser et al. [140] have shown that computing the cover polynomial is ♯P-hard. (For definitions of standard complexity classes, see for example Papadimitriou [907].) More precisely, we have the following results from [140].

Theorem 23.20. *Computing $C(D;0,0)$, $\hat{C}(D;0,0)$, $C(D;0,-1)$, $\hat{C}(D;0,-1)$, and $C(D;1,-1)$ can be done in polynomial time. For any other fixed rational numbers x and y, computing $C(D;x,y)$ is ♯P-hard with respect to polynomial-time Turing reductions, as is computing $\hat{C}(D;x,y)$.*

For the following theorem, we say that $(x,y) \in \mathbb{Q}^2$ *has a root* if there exists a digraph D such that $\hat{C}(D;x,y) = 0$.

Theorem 23.21. *Let $(x,y) \in \mathbb{Q}^2 \setminus \{(0,0),(0,-1)\}$.*

1. *If* $x \geq 0$ *and* $y = 1$*, then there exists a fully polynomial randomized approximation scheme for computing* $\hat{C}(D; x, y)$*.*

2. *If* $1 \neq y > 0$ *and* (x, y) *has a root, then* $\hat{C}(D; x, y)$ *cannot be approximated within any polynomial factor unless* RP = NP*.*

3. *If* $y \leq 0$ *and* (x, y) *has a root, then* $\hat{C}(D; x, y)$ *cannot be approximated within any polynomial factor unless* RFP = NP*.*

Here RFP is the class of all functions computable by a BPP-machine. One might expect that Theorem 23.21 also holds for the cover polynomial $C(D; x, y)$. However, except in certain special cases, this remains an open problem.

23.2.4 The path-cycle symmetric function

Chow [316] defined a symmetric function generalization of the cover polynomial called the *path-cycle symmetric function* that is analogous to the symmetric function generalization of the chromatic polynomial defined and studied by Stanley [1035] (see Section 26.3). To define it, we need some preliminaries on symmetric functions.

A finite sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of positive integers is said to be an *integer partition of* n if $\lambda_1 \geq \cdots \geq \lambda_\ell$ and the sum of the λ_i's equals n. The λ_i's are called the *parts* of λ.

Definition 23.22. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be a countably infinite sequence of commuting independent indeterminates, and let n be a positive integer. The *power sum symmetric function* p_n is the formal power series

$$p_n(\mathbf{x}) := x_1^n + x_2^n + x_3^n + \cdots . \tag{23.6}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition of n. The *power sum symmetric function* p_λ is the formal power series

$$p_\lambda(\mathbf{x}) := \prod_{i=1}^{\ell} p_{\lambda_i}(\mathbf{x}). \tag{23.7}$$

The *augmented monomial symmetric function* $\tilde{m}_\lambda$ is the formal power series

$$\tilde{m}_\lambda(\mathbf{x}) := \sum_{(i_1, \ldots, i_\ell)} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}, \tag{23.8}$$

where the sum is over all length-ℓ sequences $(i_1, \ldots, i_\ell)$ of *distinct* positive integers. If $\ell = 0$ then we set $p_\lambda(\mathbf{x}) = \tilde{m}_\lambda(\mathbf{x}) = 1$.

Definition 23.23. Let $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ be two distinct countably infinite sequences of mutually commuting independent indeterminates. Let D be a digraph. If $\mathscr{C}$ is a path-cycle cover of D, let $\lambda(\mathscr{C})$ denote the integer partition whose parts are the lengths of the paths in $\mathscr{C}$ arranged in decreasing order, and let $\mu(\mathscr{C})$ denote the integer partition whose parts are the lengths of the cycles in $\mathscr{C}$ arranged in decreasing order. The *path-cycle symmetric function* $\Xi(D;\mathbf{x},\mathbf{y})$ is the power series in $\mathbf{x}$ and $\mathbf{y}$ defined by

$$\Xi(D;\mathbf{x},\mathbf{y}) := \sum_{\mathscr{C}} \tilde{m}_{\lambda(\mathscr{C})}(\mathbf{x}) p_{\mu(\mathscr{C})}(\mathbf{y}), \tag{23.9}$$

where the sum is over all path-cycle covers $\mathscr{C}$ of D.

Example 23.24. For the digraph D in Example 23.4, $\Xi(D;\mathbf{x},\mathbf{y}) = \tilde{m}_{1,1}(\mathbf{x}) + \tilde{m}_1(\mathbf{x})p_1(\mathbf{y}) + 2\tilde{m}_2(\mathbf{x}) + p_2(\mathbf{y})$.

Note that if D is acyclic, then $\Xi(D)$ is a power series in the $\mathbf{x}$ variables only.

The connection with the cover polynomial is given by the following proposition (from [316]), whose proof is just a definition chase.

Proposition 23.25. *If i of the $\mathbf{x}$ variables and j of the $\mathbf{y}$ variables are set to 1 and the rest are set to zero, then $\Xi(D)$ becomes a finite sum that evaluates to $C(D;i,j)$.*

Unfortunately, the path-cycle symmetric function does not seem to satisfy a deletion–contraction relation. However, several facts about the cover polynomial generalize readily. The results stated in the remainder of this section are from [316].

Proposition 23.26. *If D, D_1 and D_2 are as in Corollary 23.7, then $\Xi(D;\mathbf{x},\mathbf{y}) = \Xi(D_1;\mathbf{x},\mathbf{y})\Xi(D_2;\mathbf{x},\mathbf{y})$.*

Proposition 23.27. *Let P be a finite partially ordered set. In the notation of Theorem 23.12, $\Xi(D(P);\mathbf{x},\mathbf{y}) = X(G(P);\mathbf{x})$, where X denotes Stanley's chromatic symmetric function (see Definition 26.45).*

There is also a generalization of Theorem 23.11, but its statement and proof are more involved. We need some more definitions. It is well-known [1040, Corollary 7.7.2] that the power-sum symmetric functions are algebraically independent and generate the ring of symmetric functions as a $\mathbb{Q}$-algebra.

Definition 23.28. Define an endomorphism ω on the ring of symmetric functions by setting $\omega p_n(\mathbf{x}) = -p_n(-\mathbf{x})$, where the notation "$-\mathbf{x}$" indicates that each variable x_i should be replaced by $-x_i$. Let $f_\lambda(\mathbf{x}) := \omega\tilde{m}_\lambda(\mathbf{x})$ for any integer partition λ.

Theorem 23.29. *If $\bar{D}$ is the complement of the digraph D, then*

$$\Xi(\bar{D};\mathbf{x},\mathbf{y}) = \sum_{\mathscr{C}} f_{\lambda(\mathscr{C})}(\mathbf{x}\cup\mathbf{y}) p_{\mu(\mathscr{C})}(-\mathbf{y}), \tag{23.10}$$

where the sum is over all path-cycle covers $\mathscr{C}$ of D, and $\mathbf{x}\cup\mathbf{y}$ denotes the union of all the $\mathbf{x}$ and the $\mathbf{y}$ variables.

We remark in passing that Theorems 23.11 and 23.29 are examples of *combinatorial reciprocity theorems* (theorems that give combinatorial interpretations of combinatorially defined polynomials evaluated at negative integers) and have been generalized further by Haglund [597] and Lass [759] (see also [315] and [553]).

Chow [316] proves several results about expanding $\Xi(D)$ in terms of various symmetric function bases for special digraphs. We mention just one such result.

Theorem 23.30. *If D is an acyclic digraph, then $\omega\Xi(D)$ is a nonnegative linear combination of power-sum symmetric functions.*

Note that $\omega X(G)$ is also a nonnegative linear combination of power-sum symmetric functions [1035, Corollary 2.7].

Chung and Graham [319] have generalized the cover polynomial to the *matrix cover polynomial*, which is an invariant of a matrix with elements taken from an arbitrary commutative ring with identity. The matrix cover polynomial has a symmetric function generalization that is similar in spirit to the path-cycle symmetric function.

23.3 Tutte invariants for alternating dimaps

In [480], Farr defined Tutte invariants and extended Tutte invariants for alternating dimaps, which may be thought of as a special class of digraphs equipped with additional structure.

Definition 23.31. An *alternating dimap* is a digraph with no isolated vertices, cellularly embedded in a disjoint union of oriented surfaces, where each vertex has even degree and, for each vertex v, the edges incident with v are directed alternately into and out of v, when considered in the order in which they appear around v in the embedding. An alternating dimap may have loops and/or multiple edges, and may be empty (with no vertices, edges, or faces).

Definition 23.32. In an alternating dimap, the edges around a face are consistently directed. A face is called a *c-face* or an *a-face* according to whether this direction is clockwise or anticlockwise (i.e., with the orientation or against it). Every edge lies on a c-face and also on an a-face; its *right successor* (respectively, *left successor*) is the next edge along the c-face (respectively, the a-face).

Definition 23.33. Let $\omega := \exp(2\pi i/3)$ (not to be confused with the map ω of Definition 23.28). A *1-loop* is an edge whose head has degree two. An *ω-loop* is an edge forming a single-edge a-face. An *ω^2-loop* is an edge forming a single-edge c-face. A *triloop* is an edge that is a 1-loop, an ω-loop, or an ω^2-loop. An *ultraloop* is a triloop which, together with its vertex, constitutes a connected component of the graph. (Note that an ultraloop is simultaneously a 1-loop, an ω-loop, and an ω^2-loop.) A triloop is *proper* if it is not also an ultraloop.

Next, we define the minor operations. For alternating dimaps, deleting an edge e does not usually produce an alternating dimap, but if e is an ω-loop, an ω^2-loop, or an ultraloop, then its deletion is straightforward, and is denoted by $G\backslash e$.

Definition 23.34. If G is an alternating dimap and e is an edge of G, then the *1-reduction* or *contraction* $G[1]e$ is defined as follows.

1. If the endpoints of e do not coincide, then $G[1]e$ is formed by deleting the edge e and identifying its endpoints, while preserving the order of the edges and faces around vertices.
2. If e is an ω-loop or an ω^2-loop, then $G[1]e$ is formed just by deleting e.
3. Otherwise, let v be the vertex of e, and let the edges incident with v, in cyclic order around v starting with e directed into v, be $e, a_1, b_1, \ldots, a_k, b_k, e, c_1, d_1, \ldots, c_l, d_l$. So the a_i and d_i are directed out of v while the b_i and c_i are directed into v. Replace v by two new vertices, v_1 and v_2, and reconnect the edges a_i, b_i, c_i, d_i as follows. The tail of each a_i and the head of each b_i become v_1 instead of v, while the head of each c_i and the tail of each d_i become v_2 instead of v. The edge e is deleted. The cyclic orderings of edges around v_1 and v_2 are those induced by the ordering around v.

As compensation for the absence of a true deletion operation, we have two other minor operations, ω-reduction and ω^2-reduction.

Definition 23.35. If G is an alternating dimap and e is an edge of G, then the *ω-reduction* $G[\omega]e$ (respectively, the *ω^2-reduction* $G[\omega^2]e$) is formed as follows. Let f be the left (respectively, right) successor of e, with tail v and head w. Delete e and f, and, if $e \neq f$, replace them with a new edge g from the tail of e to w. If the degree of v is two then v is deleted.

23.3.1 Simple Tutte invariants for alternating dimaps

The reduction operations give rise the following definition of a Tutte invariant for alternating dimaps.

Definition 23.36. A *simple Tutte invariant* for alternating dimaps is a function F defined on every alternating dimap such that F is invariant under isomorphism, $F(\varnothing) = 1$, and there exist w, x, y, z such that, for any alternating dimap G,

1. for any ultraloop e of G, $F(G) = wF(G\backslash e)$;
2. for any proper 1-loop e of G, $F(G) = xF(G[1]e)$;

3. for any proper ω-loop e of G, $F(G) = yF(G[\omega]e)$;

4. for any proper ω^2-loop e of G, $F(G) = zF(G[\omega^2]e)$;

5. for any edge e of G that is not an ultraloop or a triloop,

$$F(G) = F(G[1]e) + F(G[\omega]e) + F(G[\omega^2]e).$$

However, it turns out that there are not many simple Tutte invariants, as shown in [480].

Theorem 23.37. *The only simple Tutte invariants of alternating dimaps are:*

1. $F(G) = 0$ *for nonempty* G*, with* $w = 0$*;*

2. $F(G) = 3^{|E(G)|}$*, with* $w = x = y = z = 3$*;*

3. $F(G) = (-1)^{|V(G)|}$*, with* $y = z = 1$ *and* $x = w = -1$*;*

4. $F(G) = (-1)^{c(G)}$*, with* $x = z = 1$ *and* $y = w = -1$ *(where* $c(G)$ *is the number of c-faces of* G*);*

5. $F(G) = (-1)^{a(G)}$*, with* $x = y = 1$ *and* $z = w = -1$ *(where* $a(G)$ *is the number of a-faces of* G*).*

23.3.2 Extended Tutte invariants for alternating dimaps

Other definitions of Tutte invariants for alternating dimaps are possible.

Definition 23.38. Let G be an alternating dimap. A *1-semiloop* is an edge which is a loop in the underlying undirected graph of G. If e is an edge and f is its right (respectively, left) successor, then e is an *ω-semiloop* (respectively, *ω^2-semiloop*) if

1. $e = f$, or

2. $e \neq f$ and $\{e, f\}$ is a cutset of G, or

3. $e \neq f$ and deleting both e and f increases the genus of the underlying undirected graph of G.

A 1-semiloop, ω-semiloop, or ω^2-semiloop is *proper* if it is not a triloop.

Definition 23.39. An *extended Tutte invariant* for alternating dimaps is a function F defined on every alternating dimap such that F is invariant under isomorphism, $F(\varnothing) = 1$, and there exist w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l, such that, for any alternating dimap G,

1. for any ultraloop e of G, $F(G) = wF(G\backslash e)$;

2. for any proper 1-loop e of G, $F(G) = xF(G[1]e)$;

3. for any proper ω-loop e of G, $F(G) = yF(G[\omega]e)$;
4. for any proper ω^2-loop e of G, $F(G) = zF(G[\omega^2]e)$;
5. for any proper 1-semiloop e of G,
$$F(G) = aF(G[1]e) + bF(G[\omega]e) + cF(G[\omega^2]e);$$
6. for any proper ω-semiloop e of G,
$$F(G) = dF(G[1]e) + eF(G[\omega]e) + fF(G[\omega^2]e);$$
7. for any proper ω^2-semiloop e of G,
$$F(G) = gF(G[1]e) + hF(G[\omega]e) + iF(G[\omega^2]e);$$
8. for any edge e of G that is not an ultraloop or a triloop or a semiloop,
$$F(G) = jF(G[1]e) + kF(G[\omega]e) + lF(G[\omega^2]e).$$

Farr shows that the Tutte polynomial of a planar graph may be viewed as an extended Tutte invariant, as follows.

Definition 23.40. To any undirected graph G cellularly embedded in an oriented surface we can associate two alternating dimaps $\mathrm{alt}_c(G)$ and $\mathrm{alt}_a(G)$ as follows. For $\mathrm{alt}_c(G)$ (respectively, $\mathrm{alt}_a(G)$), replace each edge $e = (u, v)$ by a pair of oppositely directed edges $u \to v$ and $v \to u$, forming a clockwise (respectively, anticlockwise) face of size two. The faces of G now all correspond to anticlockwise (respectively, clockwise) faces in $\mathrm{alt}_c(G)$ (respectively, $\mathrm{alt}_a(G)$).

Theorem 23.41. *The Tutte polynomial of a plane graph G is an extended Tutte invariant of $\mathrm{alt}_c(G)$ and of $\mathrm{alt}_a(G)$.*

23.4 Gordon–Traldi polynomials

Gordon and Traldi [574, 575] introduced eight different polynomials $f_1, \ldots, f_8$ for directed graphs, each of which is some kind of analogue of the Tutte polynomial. Their general approach is to define a function r_i, for $1 \le i \le 8$, called a *rank function*, on the edges of a digraph D, and then define a corresponding Tutte-like polynomial f_i by the corank–nullity formula

$$f_i(D; t, z) := \sum_{A \subseteq E(D)} t^{r_i(E(D)) - r_i(A)} z^{|A| - r_i(A)}. \tag{23.11}$$

They also define functions c_i, and define a Tutte-like polynomial by the formula

$$f_i(D;t,z) := \sum_{A\subseteq E(D)} t^{c_i(A)-c_i(E(D))} z^{|A|+c_i(A)-|V(D)|}. \tag{23.12}$$

By setting $r_i(A) = |V(D)| - c_i(A)$, we see that Equations (23.11) and (23.12) give equivalent polynomials. For an undirected graph, we recover the Tutte polynomial by setting $r_i(A)$ to be the cardinality of the largest acyclic subset of A or $c_i(A)$ to be the number of connected components of D if the edge set is restricted to A.

Not much is known about most of the polynomials $f_1, \ldots, f_8$, other than that various specializations of them count analogues of bases, spanning sets, and independent sets. Therefore, in most cases, we limit ourselves to providing just the definitions, referring the reader to [574, 575] for further details.

23.4.1 Polynomials for rooted digraphs

Gordon and Traldi first consider *rooted digraphs*, i.e., digraphs D with a distinguished vertex $*$ called the *root*.

Definition 23.42. A subgraph T of a rooted digraph is a *$*$-rooted arborescence* if for every vertex v of T, there is a unique directed path in T from $*$ to v. A *$*$-rooted forest of arborescences* is a vertex-disjoint union of arborescences rooted at $*, v_1, v_2, \ldots$ for some vertices $v_1, v_2, \ldots$.

Definition 23.43. Let f_1, f_2 and f_3 be defined via Equation (23.11) using the following rank functions respectively:

1. $r_1(A) := \max\{|T| : T \subseteq A \text{ is a } *\text{-rooted arborescence}\}$,
2. $r_2(A) := \max\{|T \cap A| : T \subseteq E(D) \text{ is a } *\text{-rooted arborescence}\}$,
3. $r_3(A) := \max\{|F| : F \subseteq A \text{ is a } *\text{-rooted forest of arborescences}\}$.

The polynomials f_1 and f_2 can be defined for any greedoid (see Chapter 33), and f_1 in particular has been studied by Gordon and McMahon [570] and McMahon [835]. We mention here one of the main theorems of the latter.

Definition 23.44. In a rooted digraph, an edge e from u to v is a *greedoid loop* if v lies on every directed path from the root to u.

Theorem 23.45. *Let D be a rooted digraph with no greedoid loops. Then D has a directed cycle if and only if $z+1$ divides $f_1(D;t,z)$.*

23.4.2 Polynomials for unrooted digraphs

Definition 23.46. Let D a be an unrooted digraph. A set $F \subseteq E(D)$ is a *forest of rooted arborescences* if it is a vertex-disjoint union of arborescences rooted at $v_1, v_2, \ldots$ for some vertices $v_1, v_2, \ldots$. Let $r_4(A)$ be the maximum size of a forest of rooted arborescences contained in A, and let f_4 be the polynomial defined by setting $r = r_4$ in Equation (23.11).

Definition 23.47. A digraph D is *strongly connected* if, for every pair of vertices u and v, there exists a directed path from u to v. Let f_5 and f_6 be defined via Equation (23.12) using the following functions respectively:

1. $c_5(A) :=$ the number of strongly connected components of D if the edge set is restricted to A;
2. $c_6(A) := 1 +$ the smallest cardinality of a set R of (directed) edges such that $R \cup A$ strongly connects all the vertices of D. (The edges of R may or may not be edges of D.)

23.4.3 Order-dependent polynomials

Finally, we describe f_7 and f_8. In the case of f_7, we follow Gordon and Traldi by giving the recursive definition directly instead of describing r_7.

Definition 23.48. Let D be a rooted digraph, with root $*$. An edge in D is a *2-isthmus* if it is in every maximal $*$-rooted arborescence, and is a *2-loop* if it is in no maximal $*$-rooted arborescence. A 2-loop whose initial and terminal vertices coincide is called an *ordinary loop*; otherwise, a 2-loop is called a *reversed loop*.

Definition 23.49. Let D be a rooted digraph whose underlying undirected graph is connected, and let D be equipped with a total ordering O of its edges. We define a polynomial $f_7(D, O; x, y, z)$ as follows.

1. If $D = \{*\}$, then $f_7(D) = 1$.
2. Let e be the first edge in the ordering O which emanates from $*$.
   (a) $f_7(D) = xf_7(D/e)$ if e is a 2-isthmus, where D/e is the digraph obtained from D by identifying the end vertices of e and deleting e.
   (b) $f_7(D) = yf_7(D\backslash e)$ if e is an ordinary loop.
   (c) $f_7(D) = f_7(D\backslash e) + f_7(D/e)$ otherwise.
3. If no edge emanates from $*$, then let e be the first edge directed into $*$, so that e is a reversed loop. Then $f_7(D) = zf_7(D/e)$.

Definition 23.50. Let D be a rooted digraph equipped with a total ordering of its vertices. Let $r_8(A)$ be the maximum size of a subset of A that is a $*$-rooted forest of arborescences in which each arborescence is rooted at its least vertex, and let f_8 be the polynomial defined by setting $r = r_8$ in Equation (23.11).

23.5 The B-polynomial

Recently [57], Awan and Bernardi have defined a three-variable digraph polynomial that they call the *B-polynomial*. Given a function $f : V(D) \to \{1, 2, \ldots, q\}$, let $\gamma(f)$ be the number of edges $u \to v$ in $E(D)$ such that $f(v) > f(u)$, and let $\lambda(f)$ be the number of edges $u \to v$ in $E(D)$ such that $f(v) < f(u)$. Then

$$B(D; q, y, z) := \sum_{f:V(D)\to\{1,2,\ldots,q\}} y^{\gamma(f)} z^{\lambda(f)}.$$

They show that if G is an undirected graph, and D is the directed graph obtained by replacing each edge (u, v) of G with the pair of directed edges $u \to v$ and $v \to u$, then $B(D; q, y, z)$ is equivalent, up to change of variables, to the Tutte polynomial of G. In this sense, the B-polynomial generalizes the Tutte polynomial.

The B-polynomial does satisfy a certain kind of deletion-contraction recurrence, but the recurrence does not express $B(D; q, y, z)$ in terms of the B-polynomials of digraphs with fewer edges, and so it does not yield any "universality" property. However, the B-polynomial does detect several important properties of a digraph, such as acyclicity, the length of the longest directed path, and the number of strongly connected components. It also satisfies a partial planar duality relation.

The B-polynomial has a generalization to a quasisymmetric function in two sets of variables, which yields a digraph generalization of Stanley's symmetric function generalization of the Tutte polynomial. It also generalizes Ellzey's chromatic quasisymmetric function (in one set of variables) for digraphs [464], which in turn is a generalization of Shareshian and Wachs's chromatic quasisymmetric function of a graph [1001].

23.6 Open problems

1. Does there exist another definition of a digraph Tutte polynomial that is more satisfying than the ones given in this chapter?
2. Does Theorem 23.21, or something like it, hold for the cover polynomial?
3. If P is a unit interval order, is $\Xi(D(P))$ a nonnegative combination of elementary symmetric functions? This is an old conjecture of Stanley and Stembridge [1044].
4. Characterize all extended Tutte invariants for alternating dimaps. See [1182] for some recent progress on this question.