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Symplectic matroids, independent sets, and signed graphs

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Abstract

We give an independent set axiomatization of symplectic matroids, a large and important class of Coxeter matroids. As an application, we construct a new class of examples of symplectic matroids from graphs. As another application, we prove that symplectic matroids satisfy a certain "basis exchange property," and we conjecture that this basis exchange property in fact characterizes symplectic matroids.

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1. Introduction

Gelfand and Serganova [8] introduced the concept of a *WP-matroid* (now more commonly called a *Coxeter matroid*) in order to study strata on compact homogenous spaces. Every ordinary matroid is a Coxeter matroid, but not vice versa. More precisely, finite Coxeter matroids can be classified into four infinite families—ordinary matroids, symplectic matroids, orthogonal matroids, and dihedral matroids—plus a finite number of sporadic examples.

Ordinary matroids enjoy various "cryptomorphisms" or equivalent axiomatizations, e.g., in terms of independent sets, basis exchange, and circuits. They are also closely related to graphs, which provide a rich source of examples for matroid theory.

It is natural to ask if there are analogous results for Coxeter matroids. In general the answer is unknown. The present paper provides partial results for the case of symplectic

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matroids. We prove an independent set axiomatization for symplectic matroids. As an application of this result, we show how every finite undirected multigraph gives rise to a symplectic matroid. The construction makes use of signed graphs. As another application, we prove that the bases of symplectic matroids satisfy a basis exchange property, and we conjecture that this property in fact characterizes symplectic matroids.

I believe that what I have just said is sufficient motivation for this paper, but some readers may not agree with my opinion that Coxeter matroids are natural objects and interesting in their own right, and may wish to see more motivation, in the form of solutions to problems outside of Coxeter matroid theory proper. Unfortunately, the present paper does not provide this, but my belief is that it is a step in that direction, because it establishes a connection with signed graphs and also makes the subject more concrete and accessible to matroid theorists with no background in Coxeter groups.

The following notation is used throughout the paper. If S is a set, then $|S|$ denotes the cardinality of S. A singleton set $\{x\}$ is often abbreviated to x. The symbol $\backslash$ denotes set subtraction. Expressions such as $A\cup B\backslash C$ or $A\backslash B\cup C$ should be read from left to right, i.e., perform the operation involving A and B first, and then apply the operation involving C. We refer the reader to [13] for definitions of any unexplained terminology from matroid theory.

2. Symplectic matroids

The goal of this section is to give the definition of a symplectic matroid. The standard definition involves the Bruhat order on parabolic quotients of the Weyl group C_n, but in order to keep everything as simple as possible, we will take advantage of the results in [2] and define symplectic matroids in a way that requires no explicit mention of such concepts.

Let $E_{\pm n}$ be the set $\{\pm 1, \pm 2, \pm 3, \ldots, \pm n\}$. (In fact, it is not important that $E_{\pm n}$ is a set of integers; the only structure we really need is the fixed-point-free involution $x\mapsto -x$. However, we shall use integers because it simplifies notation and because the natural ordering of the integers will turn out to be convenient.) For brevity we will sometimes write the minus sign on top; e.g., we will write $\bar{1}$ for -1. If π is a permutation of $E_{\pm n}$ and $B=\{b_1,b_2,\ldots,b_k\}$ is a subset of $E_{\pm n}$, then we define

$$\pi B \stackrel{\text{def}}{=} \{\pi b_1, \pi b_2, \ldots, \pi b_k\}.$$

An important concept in Coxeter matroid theory is *admissibility*. If $S\subseteq E_{\pm n}$, define

$$\bar{S} \stackrel{\text{def}}{=} \{-s \mid s\in S\}.$$

We say that S is *admissible* if $S\cap\bar{S}=\emptyset$. A permutation π of $E_{\pm n}$ is *admissible* if $\pi(-x)=-\pi x$ for all $x\in E_{\pm n}$. A total ordering $\prec$ of the elements of $E_{\pm n}$ is *admissible* if there exists an admissible permutation π such that $x\prec y$ if and only if $\pi x<\pi y$. (The reader may find it helpful to visualize an admissible ordering as a signed permutation σ of $\{1,2,\ldots,n\}$ followed by the negative of the reversal of σ, e.g., $\bar{2},1,3,\bar{3},\bar{1},2$.) If $\prec$

is an admissible total ordering of $E_{\pm n}$, then a map $w: E_{\pm n} \to \mathbb{R}$ is said to be a *weight function compatible with* $\prec$ if $i \prec j$ implies $w(i) \leqslant w(j)$.

One way of defining ordinary matroids involves the greedy algorithm [15, Section 1.8]. This is the approach we shall take to symplectic matroids. Suppose we are given an admissible total ordering $\prec$, a weight function w compatible with $\prec$, and a collection $\mathscr{B}$ of subsets of $E_{\pm n}$. Then we define the *greedy solution* of $\mathscr{B}$ to be the element $B \in \mathscr{B}$ that is constructed as follows: We begin with no elements in B and then we consider each element of $E_{\pm n}$ in turn from the largest (relative to $\prec$) to the smallest, adding it to B unless doing so would make it impossible to end up with an member of $\mathscr{B}$ no matter which other elements of $E_{\pm n}$ we subsequently add to B.

For example, suppose $n = 3$ and our admissible total ordering is the natural order on the integers. Let $\mathscr{B} = \{\{\bar{2}, \bar{1}, 3\}, \{\bar{2}, 1, 3\}\}$. We begin by putting 3 into B, because there are certainly members of $\mathscr{B}$ containing 3. We next consider 2, but we cannot add 2 to B, because if we do then regardless of what further numbers we add to B, we can never produce a member of $\mathscr{B}$. In other words, $\{2, 3\}$ is not a subset of any member of $\mathscr{B}$. Continuing in this way, we find that the greedy solution is $\{\bar{2}, 1, 3\}$.

Finally, we say that the greedy solution B of $\mathscr{B}$ is *optimal* if $w(B) \geqslant w(B')$ for all $B' \in \mathscr{B}$, where as usual $w(B)$ denotes $\sum_{b \in B} w(b)$. We can now define a symplectic matroid.

Definition. A *symplectic matroid* is a pair $(E_{\pm n}, \mathscr{B})$ where $\mathscr{B}$ is a nonempty family of equinumerous admissible subsets of $E_{\pm n}$ with the property that for every admissible total ordering $\prec$ of $E_{\pm n}$ and every weight function compatible with $\prec$, the greedy solution of $\mathscr{B}$ is optimal. The family $\mathscr{B}$ is called the family of *bases* of the symplectic matroid.

The equivalence of this definition of symplectic matroid with the usual definition is the content of [2, Theorem 16].

An example of a symplectic matroid is $(E_{\pm 3}, \mathscr{B})$ where $\mathscr{B} = \{1\bar{3}, 2\bar{3}, \bar{1}2, \bar{1}3\}$. Here $1\bar{3}$ is to be understood as shorthand for the set $\{1, \bar{3}\}$. Note that a symplectic matroid is *not* a matroid; it is an *analogue* of a matroid. There is a sense in which ordinary matroids may be regarded as special cases of symplectic matroids, but this need not concern us here.

3. Independent sets

If $(E_{\pm n}, \mathscr{B})$ is a symplectic matroid, we define its *family* $\mathscr{I}$ *of independent sets* by

$$\mathscr{I} \stackrel{\text{def}}{=} \{I \subseteq E_{\pm n} \mid I \subseteq B \text{ for some } B \in \mathscr{B}\}. \tag{3.1}$$

In the example of a symplectic matroid given in the last section, the family of independent sets is $\mathscr{I} = \{\emptyset, 1, \bar{1}, 2, 3, \bar{3}\} \cup \mathscr{B}$. Notice that we can recover $\mathscr{B}$ from $\mathscr{I}$; the members of $\mathscr{B}$ are just the maximal members of $\mathscr{I}$ with respect to inclusion. Thus, a characterization of $\mathscr{I}$ could be used as an alternative definition or axiomatization of a symplectic matroid. This is precisely what the following theorem provides.

Theorem 1. *A subset-closed family $\mathscr{I}$ of admissible subsets of $E_{\pm n}$ is the family of independent sets of a symplectic matroid if and only if it has the following "augmentation property":*

> *If I and J are members of $\mathscr{I}$ such that $|I|<|J|$ and such that for all $y \in J \backslash I$, the set $y \cup I$ is not in $\mathscr{I}$, then $I \cup J$ is inadmissible and there exists $x \notin I$ such that both $x \cup I$ and $\bar{x} \cup I \backslash \bar{J}$ are in $\mathscr{I}$.*

A somewhat more transparent, though more verbose, statement of Theorem 1 may be obtained by introducing the concept of a *transversal*. A transversal is an admissible subset of $E_{\pm n}$ with n elements. If T is a transversal, then we define

$$\mathscr{I} \mid T \stackrel{\text{def}}{=} \{I \cap T \mid I \in \mathscr{I}\}.$$

Then it is not hard to see that the augmentation property stated in Theorem 1 may be restated as the conjunction of two sub-properties:

1. For every transversal T, $\mathscr{I}|T$ is the family of independent sets of an ordinary matroid with ground set T, and
2. If I and J are members of $\mathscr{I}$ such that $|I|<|J|$, then either there exists $x \in J \backslash I$ such that $x \cup I \in \mathscr{I}$ or there exists $x \notin I \cup J$ such that both $x \cup I \in \mathscr{I}$ and $\bar{x} \cup I \backslash \bar{J} \in \mathscr{I}$.

We remark that the fact that part 1 here is satisfied by the family of independent sets of a symplectic matroid is already known; it is essentially [2, Theorem 14].

Now for the proof of Theorem 1.

Proof. Unless otherwise specified, the terms "larger" and "smaller" in this proof refer to the admissible total ordering $\prec$. The reader should visualize such an ordering by writing out the elements in order in a horizontal line, with the largest element first.

Sufficiency: Assume that $\mathscr{I}$ has the stated property. Call a maximal (with respect to inclusion) member of $\mathscr{I}$ a "basis" of $\mathscr{I}$. All bases of $\mathscr{I}$ are admissible, and the stated property of $\mathscr{I}$ ensures that all bases of $\mathscr{I}$ have the same number of elements. Let $\mathscr{B}$ be the collection of bases of $\mathscr{I}$. We now make the following claim, which we shall call $(*)$:

$(*)$ Let $\prec$ be an admissible ordering. Let I be a set consisting of the first i elements of $E_{\pm n}$ that are picked up by the greedy algorithm (for some $i \geqslant 0$). Let J be a member of $\mathscr{I}$ such that $i<|J|$. Then the $(i+1)$st element picked up by the greedy algorithm is no smaller than the smallest element of J.

To see this, note first that if there exists $y \in J \backslash I$ such that $y \cup I \in \mathscr{I}$, then we are done, because then the greedy algorithm will pick up either y or something larger than y, and y is trivially no smaller than the smallest element of J. Otherwise, since $\mathscr{I}$ has the stated property, there exists $x \notin I$ such that $x \cup I$ and $\bar{x} \cup I \backslash \bar{J}$ are both in $\mathscr{I}$. Moreover, $I \cup J$ is inadmissible, but each of I and J is admissible, so there exists $z \in I$ such that $\bar{z} \in J$. Choose the largest such z. Then by the maximality of z, the set S of elements of I that are larger than z is a subset of $I \backslash \bar{J}$, and therefore both $S \cup x$ and $S \cup \bar{x}$ are in $\mathscr{I}$. Now $\bar{x} \notin I$ (since $x \cup I$ is in $\mathscr{I}$ and is therefore admissible) and also $x \notin I$, so neither x nor $\bar{x}$ can be larger than z—otherwise, since both x and $\bar{x}$ are "compatible" with S, the greedy algorithm would have picked one of them (or some other element not in I that is even larger), and it did not. It follows that z appears before the "halfway point" (the point between the nth and the $(n+1)$st elements in the ordering), and that x and $\bar{x}$ both appear after z but before $\bar{z}$. Then the $(i+1)$st element picked up by the greedy algorithm must be no smaller than x, which is no smaller than $\bar{z}$, which in turn is no smaller than the smallest element of J, since $\bar{z} \in J$. This proves $(*)$.

Now let $\prec$ be an admissible ordering and let w be some weight function compatible with $\prec$. Let B be the basis of $\mathscr{I}$ chosen by the greedy algorithm. We want to show that B is optimal, so let B' be another basis. We claim that for all $i > 0$, the ith element of B is no smaller than the ith element of B'. For, given i, let I be the set consisting of the largest $i-1$ elements of B and let J be the set consisting of the largest i elements of B'. Then $|I| < |J|$, so by $(*)$, the ith element picked up by the greedy algorithm (i.e., the ith element of B) is no smaller than the smallest element (i.e., the ith element) of J. This proves the claim, which in turn shows that for all i, the weight of the ith element of B is greater than or equal to the weight of the ith element of B', so indeed B is optimal.

Necessity: Suppose that $\mathscr{I}$ is the family of independent sets of a symplectic matroid, and let I and J be members of $\mathscr{I}$ such that $|I| < |J|$ and such that for all $y \in J \backslash I$, $y \cup I \notin \mathscr{I}$. Then, as already mentioned, default [2, Theorem 14] implies that $I \cup J$ is inadmissible. The set $I \cup J$ may be partitioned into four disjoint sets W, Y, Z, and $\bar{Z}$, where W, Y, and Z are defined as follows:

$$
\begin{aligned}
W &= I \backslash \bar{J}, \\
Y &= J \backslash (I \cup \bar{I}), \\
Z &= I \cap \bar{J}.
\end{aligned}
$$

In words, Z is the subset of I whose negatives are in J, W is the rest of I, and Y is what remains in J after W, Z, and $\bar{Z}$ are removed.

Now let $X = E_{\pm n} \backslash (W \cup \bar{W} \cup Y \cup \bar{Y} \cup Z \cup \bar{Z})$. Define a "half" of X to be a maximal (with respect to inclusion) admissible subset of X. Clearly, if H is a half of X, then H and $\bar{H}$ partition X into two disjoint sets and $|H| = |\bar{H}|$. Define a "$WXYZ$ ordering" to be an admissible ordering in which the elements of W come first, then the elements of some half H of X, then the elements of Y, and then the elements of Z. (This gives us half of $E_{\pm n}$, so the ordering of the rest of $E_{\pm n}$—namely $\bar{Z}\bar{Y}\bar{H}\bar{W}$—is determined.)

Now suppose we are given a $WXYZ$ ordering with the weight function that equals one on W, H, Y, Z, and $\bar{Z}$ and equals zero after that. The greedy algorithm will begin by picking up the elements of W. We claim that some element of $H \cup Y$ must be picked up after that. For if not, the algorithm will pick up Z, since these are just the remaining elements of I. Then it will skip over $\bar{Z}$. This implies that the weight of the basis chosen will be $|I|$, but J is contained in $W \cup Y \cup \bar{Z}$ so the weight of J is $|J| > |I|$, a contradiction.

The argument just given applies regardless of how the half H of X is chosen. Therefore the following set S is nonempty:

$$S = \{x \in X \mid x \cup W \in \mathscr{I} \text{ and } \bar{x} \cup W \in \mathscr{I}\} \cup \{y \in Y \mid y \cup W \in \mathscr{I}\}.$$

(For if not, we could choose a half H of X such that for *all* $x \in H$, $x \cup W$ would not be in $\mathscr{I}$, and this would cause trouble for the greedy algorithm as just explained.) Now construct an admissible ordering $\prec$ as follows. Begin with a $WXYZ$ ordering that minimizes the number of $x \in H$ such that $x \cup W \in \mathscr{I}$. Then reposition every element in $(H \cup Y) \cap S$ so that they now come after Z (but before $\bar{Z}$). Finally, reposition the "mirror images" of the elements just moved to restore admissibility. For example, if the $WXYZ$ ordering were

$$\underbrace{a\ b}_{W}\ \underbrace{c\ d}_{H}\ \underbrace{e\ f}_{Y}\ \underbrace{\ g\ }_{Z}\ \bar{g}\ \bar{f}\ \bar{e}\ \bar{d}\ \bar{c}\ \bar{b}\ \bar{a}$$

and d and e were in S but c and f were not, then $\prec$ would be given by

$$a \succ b \succ c \succ f \succ g \succ d \succ e \succ \bar{e} \succ \bar{d} \succ \bar{g} \succ \bar{f} \succ \bar{c} \succ \bar{b} \succ \bar{a}.$$

Observe that by the minimality in the choice of H, the elements $x \in H \cup Y$ that are not repositioned have the property that $x \cup W \notin H$. Now give every element up to the end of $\bar{Z}$ weight one and give the rest of the elements weight zero. The greedy algorithm applied to this ordering will pick up the elements of W, and will skip over the elements of $H \cup Y$. Then it will pick up the elements of Z, since (as before) these are just the remaining elements of I. Now, as before, J has greater weight than I, so the greedy algorithm must pick up another element before it reaches the end of $\bar{Z}$. It cannot pick up any element of $\bar{Z}$, so it must pick up one of the repositioned elements (d, e, $\bar{e}$, or $\bar{d}$ in the example above). Let x be the first element so picked up. If $x \in X$, then we see that it satisfies the desired conditions (that both $x \cup I$ and $\bar{x} \cup I \backslash \bar{J}$ are in $\mathscr{I}$). Otherwise, x cannot be in Y, because $Y \subseteq J$ and for no $x \in J \backslash I$ can we have $x \cup I \in \mathscr{I}$. So $x \in \bar{Y}$. In particular, $x \in \bar{J}$, so $\bar{x} \cup I \backslash \bar{J} \subseteq I$, which is trivially in $\mathscr{I}$. This completes the proof. □

4. From graphs to symplectic matroids

By a *graph* we mean a finite undirected multigraph. In this section we apply Theorem 1 to show how every graph gives rise to a symplectic matroid.

Let G be a graph with n edges $e_1, e_2, \ldots, e_n$. We define a family $\mathscr{I}(G)$ of admissible subsets of $E_{\pm n}$ as follows. If $S \subseteq E_{\pm n}$ is admissible, define

$$G(S) \stackrel{\text{def}}{=} \{e_i \mid i \in S \text{ or } \bar{i} \in S\}.$$

We may think of $G(S)$ as a spanning subgraph of G. We let an admissible set S be a member of $\mathscr{I}(G)$ if and only if every connected component of $G(S)$ is either a tree or satisfies the following property: It is unicyclic (i.e., a tree plus an edge), and there is an odd number of edges e_i in the unique cycle such that $\bar{i} \in S$.

Theorem 2. *For every graph G, $\mathscr{I}(G)$ is the family of independent sets of a symplectic matroid.*

The proof of Theorem 2 makes use of certain facts from [15], which we review now. A *signed graph* is a graph with a plus sign or a minus sign on each edge. A cycle in a signed graph is *balanced* if the product of the signs of its edges is positive and is *unbalanced* otherwise. Every signed graph Γ gives rise to an ordinary matroid $M(\Gamma)$, called the *bias matroid* of Γ, as follows. The ground set of $M(\Gamma)$ is the edge set of Γ (including the signs), and a set of edges is independent if every connected component is either a tree or a unicyclic graph whose unique cycle is unbalanced. That $M(\Gamma)$ is indeed a matroid is the content of [15, Theorem 5.1]. Notice that a basis of $M(\Gamma)$ cannot have more elements than G has vertices.

Comparing the language of signed graphs with our definitions of $G(S)$ and $\mathscr{I}(G)$, we see that informally speaking, our construction of $\mathscr{I}(G)$ amounts to taking the union of all $M(\Gamma)$ as Γ ranges over all 2^n signed graphs with underlying graph G. Let us now prove Theorem 2.

Proof. We apply Theorem 1, using the "verbose" form of the augmentation property. It is clear from the construction of $\mathscr{I}(G)$ that it is subset-closed. If T is a transversal, then let Γ be the signed graph that has G as its underlying graph and that has a plus or minus sign on e_i according to whether $i \in T$ or $\bar{i} \in T$. Then $\mathscr{I}(G)|T$ is equivalent to the family of independent sets of $M(\Gamma)$, and therefore yields an ordinary matroid, by [15, Theorem 5.1].

So let us suppose now that we have two members I and J of $\mathscr{I}(G)$ with $|I| < |J|$. We may assume that G is connected, since if G is disconnected, then there will be at least one connected component H such that I has fewer elements than J when we restrict to H. If the augmentation property is satisfied when we restrict to H, then it is easy to see that it is satisfied in G (essentially because membership in $\mathscr{I}(G)$ depends only on "local" structure, i.e., local to a connected component).

Consider first the case in which the spanning subgraph $G(I)$ of G has more than one connected component. Not every component of $G(I)$ can be unicyclic, for if that were the case, then $G(I)$ would have as many edges as G has vertices, and then J, by virtue of having *more* elements than I does, could not be in $\mathscr{I}(G)$. Since G is connected, this means that there must exist an edge $e_i \in G$ that connects an acyclic connected component of $G(I)$ to some other connected component of $G(I)$. Then neither i nor $\bar{i}$ is in I. Moreover, since connecting a tree to a tree yields a tree and connecting a tree to a unicyclic graph yields a unicyclic graph with the same unique cycle, it follows that both $i \cup I$ and $\bar{i} \cup I$—and therefore *a fortiori* $\bar{i} \cup I \backslash \bar{J}$—are in $\mathscr{I}(G)$. This settles the case in which $G(I)$ has more than one connected component.

So consider now the case in which $G(I)$ is connected. Then $G(I)$ cannot contain a cycle (otherwise, arguing as before, J would have more edges than is possible for an element of $\mathscr{I}(G)$), so in fact $G(I)$ is a spanning tree of G. The set $I \backslash \bar{J} \cup J$ is admissible and therefore contained in some transversal; fix any such transversal and call it T. Let I' be the set obtained from I by reversing the signs of all the elements of $I \cap \bar{J}$. Then $I' \subseteq T$, and moreover $I' \in \mathscr{I}(G)$ because $G(I') = G(I)$ is a tree. Again by [15, Theorem 5.1], $\mathscr{I}(G) \,|\, T$ yields an ordinary matroid. So since $|I'| = |I| < |J|$, there exists $y \in J \backslash I'$ such that $y \cup I' \in \mathscr{I}(G)$. Since $G(I')$ is a spanning tree, $G(y \cup I')$ is unicyclic; let C be its unique cycle. Let C_d be the subset of C on which I and J "disagree," i.e., let

$$C_d = \{e_i \in C \,|\, i \in I \text{ and } \bar{i} \in J\} \cup \{e_i \in C \,|\, i \in J \text{ and } \bar{i} \in I\}.$$

If $C_d = \emptyset$, then $y \cup I \in \mathscr{I}(G)$—because then the edges $e_i \in C$ for which $\bar{i} \in y \cup I$ coincide with the edges $e_i \in C$ for which $\bar{i} \in y \cup I'$, of which there is an odd number since $y \cup I' \in \mathscr{I}(G)$—and we are done. Otherwise, $C_d \neq \emptyset$, in which case let $x = y$ or $x = \bar{y}$, whichever choice creates an odd number of edges $e_i \in C$ with $\bar{i} \in x \cup I$. This forces $x \cup I$ to be in $\mathscr{I}(G)$, and furthermore $\bar{x} \cup I \backslash \bar{J} \in \mathscr{I}(G)$ because $G(\bar{x} \cup I \backslash \bar{J})$ is acyclic: The nonemptyness of C_d means that $G(\bar{x} \cup I \backslash \bar{J})$ is missing at least one edge of C. So we are done in this case as well. □

Those who are familiar with the concept of representability of symplectic matroids may wonder whether the above examples are representable. Neil White (personal communication) has informed me that the answer is no, and that a counterexample is given by a graph with three vertices and four edges, in which two of the edges comprise a double edge.

One nice feature of the above "graphic symplectic matroids" is that it is easy to define a deletion operation; simply delete an edge of the graph. (For general symplectic matroids this naïve deletion procedure does not always produce another symplectic matroid.) Perhaps this means that some theorems about ordinary matroids that are proved by deletion-contraction can be carried over to graphic symplectic matroids even if they do not hold for arbitrary symplectic matroids.

5. Basis exchange

Once one has an independent set axiomatization, it is natural to ask if it can be converted to a basis exchange axiom. The answer is unknown, but here is one possibility.

Conjecture 1. Let $\mathscr{B}$ be a nonempty family of equinumerous admissible subsets of $E_{\pm n}$ and let $\mathscr{I}$ be defined as in (3.1). Then $(E_{\pm n}, \mathscr{B})$ is a symplectic matroid if and only if

1. For every transversal T, $\mathscr{I}|T$ is the family of independent sets of an ordinary matroid with ground set T, and
2. For every pair of distinct members B and B' of $\mathscr{B}$ and every $x \in B$, either there exists $y \in B'$ such that $B \backslash x \cup y \in \mathscr{B}$, or there exists $y \notin B'$ and $S \subseteq B'$ such that both $B \backslash x \cup y \in \mathscr{B}$ and $B \backslash (x \cup \overline{B'}) \cup (\bar{y} \cup S) \in \mathscr{B}$.

The "only if" direction of this conjecture is easy to prove, once Theorem 1 is available. Assume that $(E_{\pm n}, \mathscr{B})$ is a symplectic matroid. Then part 1 is known, so we need only consider part 2. Let B and B' be any members of $\mathscr{B}$ and let x be any element of B. Let $I = B\backslash x$ and let $J = B'$. Then I and J are independent sets with $|I| < |J|$. Therefore, either there exists $y \in J\backslash I$ such that $y \cup I$ is independent, or there exists $y \notin I \cup J$ such that both $y \cup I$ and $I\backslash \bar{J} \cup \bar{y}$ are independent. In the former case, $y \cup I = B\backslash x \cup y$ is independent and, because $y \notin I$, it has the same number of elements as B, and therefore is a basis, so we are done. In the latter case, $y \cup I = B\backslash x \cup y$ is a basis by the same reasoning as before, so it remains to find an appropriate $S \subseteq B'$. Let

$$K \stackrel{\text{def}}{=} I\backslash \bar{J} \cup \bar{y} = B\backslash(x \cup \overline{B'}) \cup \bar{y}.$$

Then K is independent, so if it has the same number of elements as B then we may set $S = \emptyset$ and be done. Otherwise, note that $K \cup J$ is admissible, for $I\backslash \bar{J}$ certainly contains no elements of $\bar{J}$, and $\bar{y}$ is not in $\bar{J}$ either since y was chosen to be not in J. Therefore there exists a transversal containing $K \cup J$, and since $|J| = |B'| = |B| > |K|$, by part 1 there must exist $S \subseteq J = B'$ such that $K \cup S$ is a basis.

The converse is not so straightforward because there might exist independent sets I and J such that every basis B' containing J also intersects $\bar{I}$, so that naïvely extending I and J to bases and applying the basis exchange property might (undesirably) remove elements of $I\backslash \bar{J}$ from I. Of course, this might mean that Conjecture 1 is false, but if so, I believe that it should only take a small modification to make it work.

Appendix A. A note on duality

Robin Thomas once asked me if there is a notion of duality for symplectic matroids. It is possible to argue (we omit the details) that the duality of ordinary matroids stems from the existence of a diagram automorphism of the Dynkin diagram of A_n. Unfortunately, there is no diagram automorphism of C_n, and the diagram automorphism of D_n does not appear to be as interesting as the diagram automorphism of A_n. Thus, at present, it seems advisable to follow Alexandre Borovik's suggestion to use the term "symplectic matroid duality" to refer to the involution $n \mapsto -n$.

Appendix B. A note on terminology

The literature contains many concepts that are similar to the concept of a symplectic matroid, and the terminology can sometimes be confusing. Here is a brief summary that may help clarify the situation.

There is one special case of a symplectic matroid that has been rediscovered independently several times. It goes by different names: "Lagrangian matroid," "symmetric matroid" [3] (not to be confused with the symmetric matroids of [9]), "Δ-matroid" [3], and "pseudomatroid" [6]. All these concepts are equivalent, and Gelfand–Serganova symplectic matroids are strictly more general than all of them, as noted in [2]. In addition, there exists something called a "metroid" [7] which is almost equivalent to a Δ-matroid, but technically it is a special case: metroids are Δ-matroids that include the empty set as a feasible set. This is proved in [5].

A concept that is earlier than any of the above is that of a "bimatroid" [10,11] or "linking system" [14]. In [7] it is shown that a bimatroid is a special case of a metroid. In [11], two concepts that are related to bimatroids are discussed: "orthogonal matroids" and "Pfaffian structures." Orthogonal matroids are special cases of bimatroids and hence (confusingly) are special cases of Gelfand–Serganova *symplectic* matroids. Gelfand–Serganova orthogonal matroids (i.e., the case $W = D_n$) may also be viewed as special cases of Gelfand–Serganova symplectic matroids, but it is not immediately clear whether there is any direct connection between orthogonal matroids in the sense of [11] and orthogonal matroids in the sense of Gelfand–Serganova. Note that Pfaffian structures are sometimes referred to as "symplectic matroids" [10–12], but it is not immediately clear what the precise relationship between them and the other concepts mentioned above is. One can get a Pfaffian structure out of a bimatroid, but they do not seem to be strictly equivalent, and thus a Pfaffian structure does not seem to be a special case of (say) a Δ-matroid.

Finally, we mention two other concepts that might superficially appear to be related to Coxeter matroids: "Coxeteroids" [1] and "multimatroids" [4]. The definition of a Coxeteroid is motivated by the observation that matroids and Coxeter groups both satisfy an "exchange condition." A multimatroid is a certain generalization of a Δ-matroid. In neither case, however, does there seem to be more than a superficial similarity to Coxeter matroids.

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