ADVANCES IN MATHEMATICS **118**, 71–98 (1996)
ARTICLE NO. 0018

The Path-Cycle Symmetric Function of a Digraph

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Received December 12, 1994

1. INTRODUCTION

Recently, Stanley [21] has defined a symmetric function generalization of the chromatic polynomial of a graph. Independently, Chung and Graham [4] have defined a digraph polynomial called the *cover polynomial* which is closely related to the chromatic polynomial of a graph (in fact, as we shall see, the cover polynomial of a certain digraph associated to a poset P coincides with the chromatic polynomial of the incomparability graph of P) and also to rook polynomials. The starting point for the present paper is the suggestion in Chung and Graham's paper that the cover polynomial might generalize to a symmetric function in much the same way that the chromatic polynomial does.

This is indeed the case, and in this paper we shall study this symmetric function generalization of the cover polynomial. As one would expect, there are a number of generalizations and analogues of known results about the cover polynomial, the rook polynomial, and Stanley's chromatic symmetric function. In addition, however, we will obtain some unexpected dividends, such as a combinatorial reciprocity theorem that answers a question of Chung and Graham and ties together a number of known results that previously seemed unrelated ([2, Theorem 7.3], [19, Chapter 7, Theorem 2], [21, Theorem 4.2], [23, Theorem 3.2]), and a new symmetric function basis that appears to be a natural counterpart of the polynomial basis $\binom{x+n}{d}_{n=0,\ldots,d}$. One reason for these unexpected results is that this topic lies at the intersection of several branches of combinatorics; their interaction naturally gives rise to new connections and ideas. We shall indicate several directions for further research in this potentially very rich area of study.

2. DEFINITIONS AND BASIC FACTS

In this paper, the unadorned term *graph* will mean a finite simple undirected graph and the term *digraph* will mean a finite directed graph

0001-8708/96 $18.00

without multiple edges but possibly with loops and bidirected edges. If G is a graph or a digraph we let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively. If d is a positive integer, we use the notation $[d]$ for the set $\{1, 2, ..., d\}$. Note that with our conventions, a digraph D with d vertices is equivalent to a subset of $[d] \times [d]$, i.e., a *board*. (Consider the edge set of D.) We call this subset the *associated board*, and conversely given a board we call the corresponding digraph on $[d]$ the *associated digraph*.

Our notation for symmetric functions and partitions for the most part follows that of Macdonald [18], to which the reader is referred for any facts about symmetric functions that we do not explicitly reference. If τ is a set partition or an integer partition, we write $\ell(\tau)$ for the number of parts of τ, and $|\tau|$ for the sum of the sizes of the parts of τ. We define $\operatorname{sgn} \tau$ by

$$\operatorname{sgn} \tau \stackrel{\text{def}}{=} (-1)^{|\tau| - \ell(\tau)}.$$

We also define

$$r_\tau! \stackrel{\text{def}}{=} r_1!\, r_2! \cdots,$$

where r_i is the number of parts of τ of size i. We shall denote symmetric functions by a single letter such as g or by $g(x)$ if we wish to emphasize that the symmetric function is in the variables $x = (x_1, x_2, ...)$. In addition to the usual symmetric functions m_λ, p_λ, e_λ, h_λ, and s_λ, we shall need the *augmented monomial symmetric functions* $\tilde{m}_\lambda$ [5], which are defined by

$$\tilde{m}_\lambda \stackrel{\text{def}}{=} r_\lambda!\, m_\lambda.$$

We shall also need the *forgotten symmetric functions* f_λ, which are defined by

$$f_\lambda \stackrel{\text{def}}{=} (\operatorname{sgn} \lambda)\, \omega(\tilde{m}_\lambda).$$

(Warning: this is one place where we deviate from Macdonald's conventions and follow Doubilet [6] instead, since [6] contains all the results about the forgotten symmetric functions that we shall need.) The symbol ω denotes the usual involution on symmetric functions that sends e_λ to h_λ. If $g(x)$ is a symmetric function, we shall write $g(-x)$ for the function obtained by negating each variable, and we shall write $g(1^n)$ for the polynomial in the variable n obtained by setting n variables equal to one and the rest equal to zero. We will sometimes use *set* partitions instead of *integer* partitions in subscripts; for example, if π is a set partition then the expression p_π is to be understood as an abbreviation for $p_{\operatorname{type}(\pi)}$.

We shall be dealing frequently with functions in two sets of variables, so we fix some notation here. Let $\{x_1, x_2, x_3, \ldots\}$ and $\{y_1, y_2, y_3, \ldots\}$ be two sets of independent indeterminates. (Everything commutes with everything else.) An expression like $g(x, y)$ indicates that g is invariant under any permutation of the x variables and any permutation of the y variables. It is not assumed that g is necessarily invariant under permutations that mix x and y variables, except when g is one of the symmetric function bases mentioned above; in this case $p_\lambda(x, y)$ (for example) is taken to mean the power sum symmetric function in the union of the x and y variables. Expressions like $g(x, 0)$, $g(x, -y)$ and $g(1^i, 1^j)$ have their natural meanings. The notation $\omega_x g$ will indicate that for the purposes of applying ω, g is to be interpreted as a symmetric function in the x variables with coefficients in the y's.

Our first goal is to define our main object of study, the path-cycle symmetric function. As motivation we first review some material from Stanley [21] and Chung and Graham [4] that was mentioned in the introduction.

Let G be a graph and let $x_1, x_2, \ldots$ be commuting independent indeterminates. A *stable partition* of G is a partition of $V(G)$ such that no two vertices in the same block are connected by an edge. Stanley's *chromatic symmetric function* X_G is then defined by

$$X_G = X_G(x) \stackrel{\text{def}}{=} \sum_{\pi} \tilde{m}_\pi(x),$$

where the sum is over all stable partitions of G. It is easy to see that $\tilde{m}_\pi(1^i) = i^{\underline{\ell(\pi)}}$, when $X_G(1^i) = \chi_G(i)$, the chromatic polynomial of G. (We are using the notation $i^{\underline{k}} = i(i-1)\cdots(i-k+1)$ and $i^{\overline{k}} = i(i+1)\cdots(i+k-1)$.)

Now let D be a digraph. Following Chung and Graham, we say that a subset S of the edges of D is a *path-cycle cover* of D if no two elements of S lie in the same row or column of the associated board. If we think of S as a subgraph of D then we see that this condition just means that S is a (vertex-)disjoint union of directed paths and directed cycles. A path-cycle cover with no cycles is called a *path cover*. The *type* of a path-cycle cover S is the set partition of $V(D)$ where each block is the set of vertices of one of these directed paths or directed cycles. We write $\pi(S)$ for the set of blocks corresponding to the directed paths and $\sigma(S)$ for the set of blocks corresponding to the directed cycles, and we say that the type of S is (π, σ) if $\pi(S) = \pi$ and $\sigma(S) = \sigma$. Chung and Graham's *cover polynomial* $C(D; i, j)$ is then defined by

$$C(D; i, j) \stackrel{\text{def}}{=} \sum_{S} i^{\underline{\ell(\pi(S))}} j^{\ell(\sigma(S))},$$

where the sum is over all path-cycle covers $S \subset E(D)$.

In view of these definitions and the fact that $p_\sigma(1^j) = j^{\ell(\sigma)}$, the following definition (suggested by Stanley [20]) is quite natural.

DEFINITION. Let D be a digraph, and let $x = \{x_1, x_2, ...\}$ and $y = \{y_1, y_2, ...\}$ be two sets of commuting independent indeterminates. The *path-cycle symmetric function* Ξ_D of D is defined by

$$\Xi_D = \Xi_D(x, y) \stackrel{\text{def}}{=} \sum_S \tilde{m}_{\pi(S)}(x)\, p_{\sigma(S)}(y),$$

where the sum is over all path-cycle covers $S \subset E(D)$.

Note that if we only care about *path* covers we can simply consider $\Xi_D(x, 0)$. In addition, if B is the board associated with D, then $\Xi_D(0, y)$ is equivalent to what Stanley and Stembridge call $Z[B]$ ([23, Section 3]). Thus, as will become evident shortly, we may regard Ξ_D as a further generalization of Stanley and Stembridge's generalization of the theory of permutations with restricted position.

The following fact is immediate.

PROPOSITION 1. $\Xi_D(1^i, 1^j) = C(D; i, j)$.

Given a poset P, let $G(P)$ denote its incomparability graph (in which two vertices of the poset are adjacent iff they are incomparable), and let $D(P)$ denote the digraph with edge set $\{(i, j) \mid i < j\}$. Chung and Graham observed that for any poset P,

$$C(D(P); i, 0) = \chi_{G(P)}(i).$$

This connection generalizes readily to the symmetric function case.

PROPOSITION 2. *For any poset* P, $\Xi_{D(P)} = X_{G(P)}$.

Proof. Since $D(P)$ is acyclic, all path-cycle covers are in fact just path covers, so the y variables can be deleted from the definition of Ξ_D in this case. But path covers of $D(P)$ correspond to partitions of P into chains, which correspond to stable partitions of $G(P)$. Comparing the definitions of Ξ_D and X_G yields the proposition. ∎

Both the cover polynomial and the chromatic symmetric function satisfy a multiplicativity property. To prove the corresponding result for the path-cycle symmetric function it is useful to introduce the concept of a path-cycle coloring (essentially due to Chung and Graham).

DEFINITION. A *path-cycle coloring* of a digraph D is an ordered pair (S, κ) where S is a path-cycle cover and κ is a coloring of the vertices (with positive integers as colors) that is monochromatic on each path and cycle of S and which assigns distinct colors to distinct paths. A *path coloring* is a path-cycle coloring with no cycles.

PROPOSITION 3. *For any digraph D,*

$$\Xi_D = \sum_{(S,\kappa)} \prod_{u \text{ is in a path}} x_{\kappa(u)} \prod_{v \text{ is in a cycle}} y_{\kappa(v)},$$

where the sum is over all path-cycle colorings (S, κ).

Proof. Regard the sum as a double sum: for each path-cycle cover S, sum over all "compatible" colorings κ, and then sum over all S. For each fixed S the paths and cycles may be colored independently so the sum over κ factors into a product of a symmetric function in x and a symmetric function in y. Clearly coloring the paths with distinct colors gives $\tilde{m}_{\pi(S)}(x)$ and coloring the cycles so that each cycle is monochromatic gives $p_{\sigma(S)}(y)$. ∎

PROPOSITION 4. *Suppose D is the digraph formed by joining the disjoint digraphs D_1 and D_2 with all the edges (v_1, v_2) with $v_1 \in V(D_1)$ and $v_2 \in V(D_2)$. Then $\Xi_D = \Xi_{D_1} \Xi_{D_2}$.*

Proof. A path-cycle coloring of D induces path-cycle colorings of both D_1 and D_2 by restriction. Conversely, given any path-cycle coloring of D_1 and any path-cycle coloring of D_2 there exists a unique path-cycle coloring of D inducing them: if a path in D_1 has the same color as a path in D_2, join them end to end with the appropriate edge from D_1 to D_2. The result now follows from Proposition 3. ∎

3. RECIPROCITY

The *complement* D' of a digraph D is the digraph on the same vertex set whose edges are precisely those pairs (i, j) that are not edges of D. In this section we prove one of the most striking facts about the path-cycle symmetric function; namely, a combinatorial reciprocity theorem relating Ξ_D and $\Xi_{D'}$. We shall need two change-of-basis formulas, which we shall now state.

Let Π_n denote the lattice of partitions of $[n]$ (ordered by refinement). Recall that if $\pi \leqslant \sigma$ in Π_n and r_i is the number of blocks of σ that are composed of i blocks of π, then the Möbius function satisfies

$$|\mu(\pi, \sigma)| = \prod_i (i-1)!^{r_i}.$$

(See [22, Example 3.10.4] for a proof.) Also, following Doubilet [6], define

$$\lambda(\pi, \sigma)! \stackrel{\text{def}}{=} \prod_i i!^{r_i}.$$

We then have the following change-of-basis formulas (taken from [6, Appendix 1]).

PROPOSITION 5.

$$f_\pi = \sum_{\sigma \geqslant \pi} \lambda(\pi, \sigma)!\, \tilde{m}_\sigma = \sum_{\sigma \geqslant \pi} |\mu(\pi, \sigma)|\, p_\sigma.$$

We are now ready for the main theorem of this section.

THEOREM 1. *For any digraph D,*

$$\Xi_D(x, y) = \sum_S \operatorname{sgn} \pi(S)\, f_{\pi(S)}(x, y)\, p_{\sigma(S)}(-y),$$

where the sum is over all path-cycle covers of the complement D'. Equivalently,

$$\Xi_D(x, y) = [\omega_x \Xi_{D'}(x, -y)]_{x \to (x, y)},$$

where $[g(x, y)]_{x \to (x, y)}$ means that, treating g as a symmetric function in the x's with coefficients in the y's, the set of x variables is to be replaced by the union of the x and y variables.

Proof. The equivalence of the two formulations is clear. We define a *partitioned order* of D to be a partition of $V(D)$ together with either a linear order or a cyclic order on each block. If κ is a partitioned order of D, let $\pi(\kappa)$ be the set of blocks with linear orders and let $\sigma(\kappa)$ be the set of blocks with cyclic orders. Let E_κ denote the set of ordered pairs (u, v) satisfying the following two conditions.

1. u and v are in the same block of κ and u immediately precedes v in the linear or cyclic order on the block.
2. (u, v) is not an edge of D.

Note that $E_\kappa \subset E(D')$ and that there is a natural bijection between partitioned orders κ such that $E_\kappa = \varnothing$ and path-cycle covers (given such a partitioned order, take all (u, v) satisfying condition 1 above). Now for any finite set T, the alternating sum

$$\sum_{S \subset T} (-1)^{|S|}$$

equals one if $T = \varnothing$ and is zero otherwise. Thus

$$\Xi_D = \sum_{\kappa} \tilde{m}_{\pi(\kappa)}\, p_{\sigma(\kappa)} \sum_{S \subset E_\kappa} (-1)^{|S|},$$

where the first sum is over all partitioned orders of D. We now interchange the order of summation. Observe first that all sets S that arise are path-cycle covers of D', since S is a subset of the set of all (u, v) satisfying condition 1 above for some κ. Given a path-cycle cover S of D', we now need to determine the set $\mathscr{P}$ of partitioned orders of D that give rise to it. Only blocks with cyclic orders can give rise to cycles of S, so for every $\kappa \in \mathscr{P}$, $\sigma(\kappa)$ must include the blocks of $\sigma(S)$ among its own blocks. On the other hand, the blocks of $\pi(S)$ can arise either from blocks with linear orders or from blocks with cyclic orders. To determine all possibilities we must consider all ways of agglomerating the blocks of $\pi(S)$ into blocks of $\pi(\kappa)$, and then for each composite block in each such agglomeration we must consider both linear and cyclic orders. The linear or cyclic order on the composite block can be viewed as a linear or cyclic order on the *blocks of* $\pi(S)$ (instead of on the vertices), because the linear or cyclic order must induce the edges of S, i.e., if (u, v) is an edge of S then u must immediately precede v in the order dictated by κ, and therefore the vertices in each block of $\pi(S)$ are constrained to be consecutive and in a fixed order. Clearly, every such linear or cyclic order on the blocks gives rise to a unique $\kappa \in \mathscr{P}$. The number of ways to impose a linear order if there are i blocks is $i!$ and the number of ways to impose a cyclic order is $(i-1)!$. Thus we can enumerate $\mathscr{P}$ by summing over all divisions of the blocks of $\pi(S)$ into two groups α and β (linear and cyclic) and, for each such division, summing over all ways of grouping the blocks into composite blocks, weighted by a factorial factor. More precisely we have

$$\Xi_D = \sum_{S} (-1)^{|S|}\, p_{\sigma(S)}(y) \sum_{(\alpha, \beta)} \sum_{\substack{\gamma \geqslant \alpha \\ \delta \geqslant \beta}} \lambda(\alpha, \gamma)!\, |\mu(\beta, \delta)|\, \tilde{m}_\gamma(x)\, p_\delta(y),$$

where the first sum is over all path-cycle covers of D'. By Proposition 5, we have

$$\begin{aligned}\Xi_D &= \sum_{S} (-1)^{|S|}\, p_{\sigma(S)}(y) \sum_{(\alpha, \beta)} f_\alpha(x)\, f_\beta(y)\\ &= \sum_{S} (-1)^{|S|}\, p_{\sigma(S)}(y) \sum_{(\alpha, \beta)} (\operatorname{sgn} \alpha)(\operatorname{sgn} \beta)\, \omega_x \tilde{m}_\alpha(x)\, \omega_y \tilde{m}_\beta(y).\end{aligned}$$

Now the blocks of α and β correspond to the paths of S, so $|\alpha| - \ell(\alpha)$ is the number of edges of S in α, and similarly for β. Thus $(\operatorname{sgn} \alpha)(\operatorname{sgn} \beta)$ depends only on the total number of edges of S devoted to directed paths (namely, $|\pi(S)|-\ell(\pi(S))$) and does not depend on the particular choice of α or β. We have

$$\Xi_D=\sum_S(-1)^{|S|}\,p_{\sigma(S)}(y)(-1)^{|\pi(S)|-\ell(\pi(S))}\sum_{(\alpha,\beta)}\omega_x\tilde{m}_\alpha(x)\,\omega_y\tilde{m}_\beta(y)$$
$$=\sum_S(-1)^{|\sigma(S)|}\,p_{\sigma(S)}(y)\,\omega_x\omega_y\sum_{(\alpha,\beta)}\tilde{m}_\alpha(x)\,\tilde{m}_\beta(y).$$

A moment's thought shows that the inner sum is $\tilde{m}_{\pi(S)}(x, y)$. Now

$$\omega_x\omega_y p_n(x, y)=\omega_x\omega_y(p_n(x)+p_n(y))$$
$$=(-1)^{n-1}\,p_n(x)+(-1)^{n-1}\,p_n(y)=[\omega_x p_n(x)]_{x\to(x,y)},$$

and since ω and $x\to(x, y)$ are both homomorphisms,

$$\omega_x\omega_y g(x, y)=[\omega_x g(x)]_{x\to(x,y)}$$

for any symmetric function g. Finally, $(-1)^{|\sigma|}\,p_\sigma(y)=p_\sigma(-y)$, so we obtain

$$\Xi_D=\sum_S p_{\sigma(S)}(-y)[\omega_x\tilde{m}_{\pi(S)}(x)]_{x\to(x,y)}$$

as desired. ∎

We remark that the appearance of ω in Theorem 1 is what leads us to call it a combinatorial reciprocity theorem.

Theorem 1 readily yields several attractive corollaries. For example, by setting all the x variables equal to zero, we immediately obtain [23, Theorem 3.2]. More interestingly, we can obtain an affirmative answer to the question, raised by Chung and Graham, of whether $C(D; i, j)$ determines $C(D'; i, j)$.

Corollary 1. *If D is a digraph with d vertices, then* $C(D'; i, j)=(-1)^d\,C(D; -i-j, j)$.

Proof. Let g be any symmetric function that is homogeneous of degree d and let $g^*=\omega g$. We claim that $g^*(1^i)$ is obtained by changing i to $-i$ and $g(1^i)$ and then multiplying by $(-1)^d$. To see this, first consider the case where $g=p_\lambda$ for some $\lambda\vdash d$. Then $g^*=(\operatorname{sgn}\lambda)\,p_\lambda$ and hence

$$g^*(1^i)=(\operatorname{sgn}\lambda)\,i^{\ell(\lambda)}.$$

On the other hand $g(1^i)=i^{\ell(\lambda)}$. Changing i to $-i$ and multiplying by $(-1)^d$ amounts to multiplying by $(-1)^{d-\ell(\lambda)}=\operatorname{sgn}\lambda$, as required. The claim then follows by linearity.

Now $\tilde{m}_\pi(1^i, 1^j) = (i+j)^{\underline{\ell(\pi)}}$. Since $(\operatorname{sgn} \pi)\, f_\pi = \omega \tilde{m}_\pi$, we have

$$(\operatorname{sgn} \pi)\, f_\pi(1^i, 1^j) = (-1)^{|\pi|}\, (-i-j)^{\underline{\ell(\pi)}}.$$

Also, as noted before, $p_\sigma(-y) = (-1)^{|\sigma|}\, p_\sigma(y)$. Thus, specializing Theorem 1 via Proposition 1 yields

$$\begin{aligned} C(D; i, j) &= \sum_S (-1)^{|\sigma(S)|}\, (-1)^{|\pi(S)|}\, (-i-j)^{\underline{\ell(\pi(S))}}\, j^{\ell(\sigma(S))} \\ &= (-1)^d\, C(D'; -i-j, j). \quad \blacksquare \end{aligned}$$

Corollary 1 can be proved directly using deletion-contraction techniques, and it has also been obtained independently by Gessel [9]. We omit the details.

A further specialization of Theorem 1 gives a formula for rook polynomials; we defer this to the next section, where we consider rook theory in more detail.

COROLLARY 2. *For any digraph D,*

$$\Xi_D(x, 0) = \omega_x \Xi_{D'}(x, 0).$$

Note the similarity between this result and Stanley's reciprocity theorem [21, Theorem 4.2.]. In fact, the two reciprocity theorems overlap, because of Proposition 2, so Corollary 2 gives a new interpretation of $\omega \Xi_{D(P)} = \omega \Xi_{G(P)}$ when P is a poset.

Corollary 2 follows immediately from Theorem 1, but we shall give two other proofs because they illustrate connections with other known results. The first proof is due to Gessel [9], and it derives Corollary 2 from a result of Carlitz, Scoville and Vaughan [2, Theorem 7.3]. We need some preliminaries. Given a digraph D with d vertices, let

$$A_D = \{a_1, a_2, ..., a_d\}$$

be a set of commuting independent indeterminates, and define

$$\alpha_{D,n} = \sum_{i_1, i_2, ..., i_n} a_{i_1} a_{i_2} \cdots a_{i_n},$$

where the sum is over all $i_1, i_2, ..., i_n$ such that $(a_{i_j}, a_{i_{j+1}})$ is an edge of D for all $j < n$. Similarly, let

$$\alpha'_{D,n} = \sum_{i_1, i_2, ..., i_n} a_{i_1} a_{i_2} \cdots a_{i_n},$$

where this time the sum is over all $i_1, i_2, ..., i_n$ such that $(a_{i_j}, a_{i_{j+1}})$ is an edge of the complement D' for all $j<n$. With this notation, the result of Carlitz, Scoville and Vaughan is (essentially) the following.

PROPOSITION 6. *For any digraph D,*

$$\sum_n (-1)^n \alpha'_{D,n} = \left(\sum_n \alpha_{D,n}\right)^{-1}.$$

We can now give Gessel's proof of Corollary 2.

First Proof of Corollary 2. Let $\theta_{D,y}$ be the homomorphism from the ring of symmetric functions in the variables $y = \{y_1, y_2, ...\}$ to the ring of formal power series in A_D that sends the complete symmetric function $h_n(y)$ to $\alpha_{D,n}$. Similarly, let $\theta'_{D,y}$ be the homomorphism that sends $h_n(y)$ to $\alpha'_{D,n}$. From [18, (4.2)] we have

$$\prod_{i,j} \frac{1}{1-x_i y_i} = \sum_\lambda h_\lambda(y)\, m_\lambda(x).$$

Applying $\theta_{D,y}$ gives

$$\theta_{D,y}\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right) = \sum_\lambda \alpha_{D,\lambda_1}\alpha_{D,\lambda_2}\cdots m_\lambda(x).$$

Now $\Xi_D(x, 0)$ is just the coefficient of $a_1 a_2 \cdots a_d$ in this expression, since this coefficient counts all path covers of type π exactly $r_\pi!$ times, and $r_\pi!\, m_\pi(x) = \tilde{m}_\pi(x)$. Similarly, $\Xi_{D'}(x, 0)$ is the coefficient of $a_1 a_2 \cdots a_d$ in

$$\theta'_{D,y}\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right) = \sum_\lambda \alpha'_{D,\lambda_1}\alpha'_{D,\lambda_2}\cdots m_\lambda(x).$$

Thus it suffices to prove that

$$\omega_x \theta_{D,y}\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right) = \theta'_{D,y}\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right).$$

Now from [18, (4.3)] we have

$$\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right) = \sum_\lambda s_\lambda(x)\, s_\lambda(y),$$

so

$$\omega_x\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right)=\sum_\lambda s_{\lambda'}(x)\,s_\lambda(y)=\sum_\lambda s_\lambda(x)\,s_{\lambda'}(y)=\omega_y\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right).$$

Thus

$$\omega_x\theta_{D,y}\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right)=\theta_{D,y}\omega_x\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right)=\theta_{D,y}\omega_y\left(\prod_{i,j}\frac{1}{1-x_iy_j}\right).$$

So it suffices to show that $\theta_{D,y}\omega_y=\theta'_{D,y}$. From [18, (2.6)] we have

$$\sum_n(-1)^n\,e_n(y)=\left(\sum_n h_n(y)\right)^{-1},$$

so applying $\theta_{D,y}$ and using Proposition 6 yields

$$\sum_n(-1)^n\,\theta_{D,y}(e_n(y))=\left(\sum_n \alpha_{D,n}\right)^{-1}=\sum_n(-1)^n\,\alpha'_{D,n}.$$

Equating terms of the same degree, we see that

$$\theta_{D,y}\omega_y(h_n(y))=\theta_{D,y}(e_n(y))=\alpha'_{D,n}=\theta'_{D,y}(h_n(y)),$$

completing the proof. ∎

Our second proof of Corollary 2 is similar to Stanley's proof of the reciprocity theorem for X_G. Following Gessel [10] and Stanley [21, Section 3], we define a power series in the variables $x=\{x_1,x_2,...\}$ to be *quasi-symmetric* if the coefficients of

$$x_{i_1}^{r_1}x_{i_2}^{r_2}\cdots x_{i_k}^{r_k}\qquad\text{and}\qquad x_{j_1}^{r_1}x_{j_2}^{r_2}\cdots x_{j_k}^{r_k}$$

are equal whenever $i_1<i_2<\cdots<i_k$ and $j_1<j_2<\cdots<j_k$. For any subset S of $[d-1]$ define the *fundamental* quasi-symmetric function $Q_{S,d}(x)$ by

$$Q_{S,d}(x)=\sum_{\substack{i_1\leqslant\cdots\leqslant i_d\\ i_j<i_{j+1}\text{ if } j\in S}}x_{i_1}x_{i_2}\cdots x_{i_d}.$$

We have the following expansion of $\Xi_D(x,0)$ in terms of fundamental quasi-symmetric functions.

Proposition 7. *If D is a digraph with vertex set $[d]$, then*

$$\Xi_D(x,0)=\sum_{\pi\in S_d}Q_{S(\pi),d}(x),$$

where S_d is the group of permutations of $[d]$ and

$$S(\pi)=\{i\in[d] \mid (\pi_i, \pi_{i+1}) \text{ is not an edge of } D\}.$$

Proof. We use the expression for Ξ_D given in Proposition 3. Given a path coloring of D, arrange the paths in increasing order of their colors, and within each path arrange the vertices in the order given by the directed path. This gives a permutation of the vertices of D, and it is easy to see that $Q_{S(\pi),d}(x)$ counts precisely the path colorings that give rise to π. ∎

We can now give our second proof of Corollary 2.

Second Proof of Corollary 2. Without loss of generality we may assume that the vertex set of D is $[d]$. From the same argument as in Proposition 7, we see that

$$\Xi_{D'}(x,0)=\sum_{\pi\in S_d} Q_{[d]\backslash S(\pi),d}(x).$$

In view of Proposition 7, it suffices to show that the map that sends $Q_{S,d}$ to $Q_{[d]\backslash S,d}$ equals ω when restricted to symmetric functions. This is proved by Stanley in the course of proving his reciprocity theorem [21, Theorem 4.2.]. ∎

Stanley [21] has obtained an analogue of Proposition 7 by using the theory of acyclic orientations and P-partitions. It is natural to ask if these ideas can be applied to studying Ξ_D. Unfortunately this does not seem possible. For example, a key step in the proof of the analogue of Proposition 7 involves expressing X_G as a sum of certain poset generating functions, but in general Ξ_D has no such expression.

4. Rook Theory

Let $B\subset[d]\times[d]$ be a board, and let the *rook number* r_k^B denote the number of ways of placing k non-taking rooks on B. Following Goldman, Joichi and White [12], we define the *factorial polynomial* $R(B;i)$ by

$$R(B;i)\stackrel{\text{def}}{=}\sum_k r_k^B i^{\underline{d-k}}.$$

If D is the digraph associated with B, we also write r_k^D for r_k^B and $R(D;i)$ for $R(B;i)$. (With this equivalence between boards and digraphs, the factorial polynomial is the same as Chung and Graham's *binomial drop polynomial.*) The study of the factorial polynomial and other rook polynomials is a well-established area of combinatories (see for example Riordan [19, Chapter 7 and 8] for a classical account and Goldman, Joichi, Reiner and White's papers [12, 13, 14, 15, 16] for further results). The definition of a path-cycle cover already suggests a connection with rook theory. More precisely, we have the following observation of Chung and Graham.

PROPOSITION 8. *For any digraph* D, $R(D; i) = C(D; i, 1) = \Xi_D(1^i, 1)$.

Proof. The first equality is demonstrated in Chung and Graham [4] and the second equality follows from Proposition 1. ∎

Proposition 8 (as well as, for example, Stanley and Stembridge [23, Section 3] and Stanley [21, Proposition 5.5]) suggests that some of the theory of rook polynomials might generalize to Ξ_D. This is indeed the case, as we shall now see. In fact, we have already seen an example in the last section, since Theorem 1 can be viewed as a generalization of a result in Riordan [19, Chapter 7, Theorem 2] relating the rook numbers of complementary boards, a result which we now state. If B is a board, we let $B' = ([d] \times [d]) \backslash B$ denote the complementary board.

PROPOSITION 9. *Let* $B \subset [d] \times [d]$ *be a board. Then* $R(B'; i) = (-1)^d R(B; -i-1)$.

Proof. Let D be the associated digraph. From Corollary 1 and Proposition 8 we have

$$\begin{aligned} R(B'; i) &= C(D'; i, 1) = (-1)^d\, C(D; -i-1, 1) \\ &= (-1)^d\, R(B; -i-1). \quad ∎ \end{aligned}$$

Riordan's original result is

$$\sum_k r_k^B (d-k)!\, i^k = \sum_k (-1)^k\, r_k^{B'} (d-k)!\, i^k (i+1)^{d-k},$$

which can be shown to be equivalent to Proposition 9. However, it seems that our formulation—in particular the observation that Proposition 9 is essentially a combinatorial reciprocity theorem—is new (although as Gessel [9] has observed, it follows immediately from the next proposition below).

Our next result is a generalization of the fundamental inclusion-exclusion formula of rook theory. This formula has many equivalent formulations; we shall use the following one whose proof is given implicitly by Chung and Graham.

PROPOSITION 10. *Let D be a digraph with d vertices, and let N_k^D denote the number of ways of placing d non-taking rooks on $[d] \times [d]$ such that exactly k rooks lie on the board associated with D. Then*

$$R(D; i) = \sum_k N_k^D \binom{i+k}{d}.$$

To state our generalization we need a few more definitions.

DEFINITION. For any pair of integer partitions λ and μ, define $D_{\lambda, \mu}$ to be a disjoint union of directed paths and directed cycles such that the ith directed path has λ_i vertices and the jth directed cycle has μ_j vertices. Define

$$\tilde{\Xi}_{\lambda, \mu} \stackrel{\text{def}}{=} \sum_S \frac{\tilde{m}_{\pi(S)}(x)\, p_{\sigma(S)}(y)}{\ell(\pi(S))!},$$

where the sum is over all path-cycle covers S of $D_{\lambda, \mu}$. For brevity we shall write D_λ for $D_{\lambda, \varnothing}$ and $\tilde{\Xi}_\lambda$ for $\tilde{\Xi}_{\lambda, \varnothing}$. We then have the following result.

THEOREM 2. *Let D be a digraph with d vertices and let B be the associated board. Let $\mathcal{N}^D_{\lambda, \mu}$ be the set of placements of d non-taking rooks and $[d] \times [d]$ such that the type (π, σ) of the path-cycle cover formed by the set of edges corresponding to rooks placed on B satisfies* $\operatorname{type}(\pi) = \lambda$ *and* $\operatorname{type}(\sigma) = \mu$, *and let $N^D_{\lambda, \mu} = |\mathcal{N}^D_{\lambda, \mu}|$. Then*

$$\Xi_D = \sum_{\lambda, \mu} N^D_{\lambda, \mu} \tilde{\Xi}_{\lambda, \mu},$$

where the sum is over all integer partitions λ and μ.

Proof. The proof is similar to the proof of [22, Theorem 2.3.1]. Given any two integer partitions ν and η, let $\mathcal{R}^D_{\nu, \eta}$ be the set of path-cycle covers S of D satisfying $\operatorname{type}(\pi(S)) = \nu$ and $\operatorname{type}(\sigma(S)) = \eta$. Note that every element of $\mathcal{R}^D_{\nu, \eta}$ has $\ell(\nu)$ directed paths plus some cycles and therefore has a total of $d - \ell(\nu)$ edges.

Now fix a pair of integer partitions ν and η and consider the set of pairs (S, T) such that $S \in \mathcal{R}^D_{\nu, \eta}$ and T is an extension of S (regarded as a placement of non-taking rooks on B) to a placement of d non-taking rooks on $[d] \times [d]$. This set has $\ell(\nu)!\, |\mathcal{R}^D_{\nu, \eta}|$ elements, since for each $S \in \mathcal{R}^D_{\nu, \eta}$ the $\ell(\nu)$ rows and columns unoccupied by S can support $\ell(\nu)!$ placements of non-taking rooks. On the other hand, we can enumerate the set in another way, by taking each placement of d non-taking rooks on $[d] \times [d]$ and counting bow many $S \in \mathcal{R}^D_{\nu, \eta}$ it extends. Now if $T \in \mathcal{N}^D_{\lambda, \mu}$ for some λ and μ, then the number of elements of $\mathcal{R}^D_{\nu, \eta}$ that it extends is just the number

$n_{\lambda,\mu,\nu,\eta}$ of path-cycle covers S of $D_{\lambda,\mu}$ satisfying $\text{type}(\pi(S))=\nu$ and $\text{type}(\sigma(S))=\eta$. Since every placement of d non-taking rooks on $[d]\times[d]$ belongs to $\mathcal{N}^D_{\lambda,\mu}$ for some λ and μ, we have

$$\sum_{\lambda,\mu} N^D_{\lambda,\mu} n_{\lambda,\mu,\nu,\eta} = \ell(\nu)!\ |\mathcal{R}^D_{\nu,\eta}|.$$

Now divide both sides by $\ell(\nu)!$, multiply both sides by $\tilde{m}_\nu(x)\, p_\eta(y)$, and sum over all ν and η to obtain the desired result. ∎

The next proposition shows that Theorem 2 does indeed generalize Proposition 10.

PROPOSITION 11. *For any integer partitions λ and μ,*

$$\tilde{\tilde{\Xi}}_{\lambda,\mu}(1^i, 1) = \binom{i+d-\ell(\lambda)}{d},$$

where $d=|\lambda|+|\mu|$.

Proof. Directly from the definitions we have

$$\tilde{\tilde{\Xi}}_{\lambda,\mu}(1^i, 1) = \sum_S \frac{i^{\underline{\ell(\pi(S))}}}{\ell(\pi(S))!} = \sum_S \binom{i}{\ell(\pi(S))},$$

where the sum is over all path-cycle covers of $D_{\lambda,\mu}$. The sum can be broken up into a double sum:

$$\tilde{\tilde{\Xi}}_{\lambda,\mu}(1^i, 1) = \sum_k \sum_{\{S \mid \ell(\pi(S))=k\}} \binom{i}{k}.$$

But $\ell(\pi(S))=k$ if and only if $|S|=d-k$. Since every subset of the edges of $D_{\lambda,\mu}$ is a path-cycle cover, and since the total number of edges of $D_{\lambda,\mu}$ is $d-\ell(\lambda)$, we have

$$\tilde{\tilde{\Xi}}_{\lambda,\mu}(1^i, 1) = \sum_k \binom{d-\ell(\lambda)}{d-k}\binom{i}{k} = \binom{i+d-\ell(\lambda)}{d}. \quad ∎$$

In view of Proposition 8, Proposition 11, and the fact that the number of edges of a path-cycle cover of type (π, σ) is $d-\ell(\pi)$, we see that Theorem 2 implies Proposition 10.

It might seem that Theorem 2 is rather contrived, since we seem to have defined $\tilde{\tilde{\Xi}}_{\lambda,\mu}$ just so that Theorem 2 would come out right. In fact, however, the functions $\tilde{\tilde{\Xi}}_{\lambda,\mu}$ have intrinsic interest, as we shall now illustrate.

PROPOSITION 12. *The functions $\tilde{\tilde{\Xi}}_\lambda$ form a linear basis for the ring of symmetric functions over the rationals, and the functions $\tilde{\tilde{\Xi}}_{\lambda,\mu}$ form a linear* *basis for the ring of symmetric functions in two sets of variables* (*again over the rationals*).

Proof. Write

$$\tilde{\tilde{\Xi}}_\lambda = \sum_\mu c_{\lambda,\mu} \tilde{m}_\mu.$$

Then it is clear from the definition of $\tilde{\tilde{\Xi}}_\lambda$ that $c_{\lambda,\lambda} \neq 0$ and $c_{\lambda,\mu} \neq 0$ only if $\lambda \geqslant \mu$ in refinement order. Thus the matrix $(c_{\lambda,\mu})$ with respect to any linear extension of refinement order is triangular with nonzero entries on the diagonal. This proves the first assertion. To prove the second assertion, define a partial order on pairs of integer partitions by setting $(\lambda, \mu) < (\nu, \eta)$ if the multiset of parts of η can be partitioned into two multisets α and β such that $\mu = \beta$ and λ is a refinement of $\nu \cup \alpha$. The same kind of reasoning as before, with this partial order in place of refinement order and with the basis $\tilde{m}_\lambda(x)\, p_\mu(y)$ in place of $\tilde{m}_\lambda$, can then be applied to prove the second assertion. ∎

Unfortunately, we cannot replace "rationals" with "integers" in the above proposition, as the following table of values illustrates.

$$\begin{bmatrix} \tilde{\tilde{\Xi}}_2 \\ \tilde{\tilde{\Xi}}_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 \\ m_{11} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\tilde{\Xi}}_3 \\ \tilde{\tilde{\Xi}}_{21} \\ \tilde{\tilde{\Xi}}_{111} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_3 \\ m_{21} \\ m_{111} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\tilde{\Xi}}_4 \\ \tilde{\tilde{\Xi}}_{31} \\ \tilde{\tilde{\Xi}}_{22} \\ \tilde{\tilde{\Xi}}_{211} \\ \tilde{\tilde{\Xi}}_{1111} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{2}{3} & 1 \\ 0 & 0 & 1 & \frac{2}{3} & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_4 \\ m_{31} \\ m_{22} \\ m_{211} \\ m_{1111} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\tilde{\Xi}}_5 \\ \tilde{\tilde{\Xi}}_{41} \\ \tilde{\tilde{\Xi}}_{32} \\ \tilde{\tilde{\Xi}}_{311} \\ \tilde{\tilde{\Xi}}_{221} \\ \tilde{\tilde{\Xi}}_{2111} \\ \tilde{\tilde{\Xi}}_{11111} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & 0 & \frac{2}{3} & \frac{1}{3} & \frac{3}{4} & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{3} & \frac{2}{3} & \frac{3}{4} & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_5 \\ m_{41} \\ m_{32} \\ m_{311} \\ m_{221} \\ m_{2111} \\ m_{11111} \end{bmatrix}$$

The bases $\tilde{m}_\lambda$, p_λ, e_λ, h_λ, s_λ, and f_λ occur frequently "in nature." Similarly, there are certain "natural" bases for polynomials, and moreover there is a correspondence between some of the symmetric function bases and the polynomial bases given by $g \mapsto g(1^i)$, e.g., p_λ corresponds to i^n, and the reciprocally related bases $\tilde{m}_\lambda$ and f_λ correspond to $i^{\underline{n}}$ and $i^{\bar{n}}$. From Proposition 11 and Proposition 12 we see that the basis $\tilde{\Xi}_\lambda$ seems to be a promising candidate for the natural counterpart of the basis

$$\binom{i+n}{d}_{n=0,\,1,\,\ldots,\,d}.$$

As further evidence for this assertion we state the following generalization of the fact, essentially due to Vo [24] and Linial [17] (but see also [1] and [7]), that the expansion of the chromatic polynomial in terms of the basis $\binom{i+n}{d}$ has nonnegative integer coefficients. We will give the proof elsewhere [3] since the result is tangential to our main purpose.

THEOREM 3. *For any graph G, the expansion of X_G in terms of the basis $\tilde{\Xi}_\lambda$ has nonnegative integer coefficients.*

It is natural to ask about the connection between the basis $\tilde{\Xi}_\lambda$ and the standard symmetric function bases. We give just one result along these lines.

PROPOSITION 13. *The linear map that sends $\tilde{\Xi}_\lambda$ to $(\operatorname{sgn}\lambda)\,\tilde{m}_\lambda/\ell(\lambda)!$ is an involution.*

Proof. Given any integer partitions λ and μ, let π be any set partition of type λ and define

$$c_{\lambda,\mu} \stackrel{\text{def}}{=} \sum_{\{\sigma \geqslant \pi \mid \operatorname{type}(\sigma)=\mu\}} \lambda(\pi,\sigma)!.$$

Note that $c_{\lambda,\mu}$ does not depend on the choice of π. We claim that

$$\Xi_{D_\mu} = \sum_\lambda \frac{r_\mu!}{r_\lambda!}\, c_{\lambda,\mu} \tilde{m}_\lambda.$$

To see this, first consider the case where $r_\mu! = r_\lambda! = 1$, i.e., the case of distinct parts. We have a disjoint union D_μ of directed paths and we want to count the number of path covers of type λ. In path cover of D_μ, each directed path is broken up into a sequence of smaller directed paths. So the path covers can be enumerated as follows: take a set partition π of type λ and consider all ways of grouping its blocks into a partition σ of type μ and then linearly ordering the blocks of π within each block of σ. Such a configuration determines a path cover: for any block b of σ, the sequence of blocks of π in b dictates the sizes of the sequence of smaller directed paths composing the directed path in D_μ corresponding to b. It is easy to see that this correspondence is bijective, and this proves our claim in the case of distinct parts. For the general case, observe that we want equal-sized parts of π to be indistinguishable and equal-sized parts of σ to be distinguishable, so we must multiply by $r_\mu!/r_\lambda!$.

From Proposition 5 we see that the matrix $((\operatorname{sgn}\lambda)\, c_{\lambda,\mu})$ is the matrix of ω and is therefore an involution. From our claim it follows that the matrix relating $(\operatorname{sgn}\lambda)\, \tilde{m}_\lambda/r_\lambda!$ and $\Xi_{D_\mu}/r_\mu!$ or equivalently the matrix relating

$$\frac{(\operatorname{sgn}\lambda)\, \tilde{m}_\lambda}{r_\lambda!\, \ell(\lambda)!} \qquad \text{and} \qquad \frac{\tilde{\Xi}_\mu}{r_\mu!}$$

is an involution. But then the desired result follows, since the factors of $r_\lambda!$ and $r_\mu!$ amount to conjugating by a (diagonal) matrix, and this does not change the involution property. ∎

It is natural to ask if $\tilde{\Xi}_{\lambda,\mu}(1^i, 1^j)$ gives a natural basis for polynomials in two variables. Unfortunately, this does not seem to be true. However, the specialization $g \mapsto g(1^i, 1^j)$ applied to Theorem 2 does hive us a simple proof of a theorem of Chung and Graham whose original proof is quite complicated. Following Chung and Graham, for any placement T of d non-taking rooks on $[d]\times[d]$, let drop (T) be the subgraph of D with edges corresponding to the squares occupied by the rooks of T. If D is a digraph with d vertices, let $\delta_D(q, r, s)$ be the number of ordered pairs (S, T) such that S is a set of r edges of D forming precisely s disjoint cycles and T is a placement of d non-taking rooks on $[d]\times[d]$ with $S \subset \operatorname{drop}(T)$ and $|\operatorname{drop}(T)| = q + r$. Chung and Graham's result [4, Theorem 2] is then the following.

PROPOSITION 14. *For any digraph D with d vertices,*

$$C(D; i, j) = \sum_{q, r, s} \delta_D(q, r, s) \binom{i+q}{d-r} (j-1)^s.$$

Proof. We can restate the desired result as

$$\begin{aligned} C(D; i, j+1) &= \sum_{q, r, s, t} \delta_D(q, r, s) \binom{q}{d-r-t} \binom{i}{t} j^s \\ &= \sum_{q, r, s, t} \delta_D(q, r, s) \binom{q}{t-d+r+q} \binom{i}{t} j^s. \end{aligned}$$

From Theorem 2 and the definition of $\tilde{\Xi}_{\lambda,\mu}$ we have

$$C(D; i, j+1) = \sum_{\lambda,\mu,t,u} N^D_{\lambda,\mu} n_{\lambda,\mu,t,u} \binom{i}{t} (j+1)^u$$

$$= \sum_{\lambda,\mu,s,t,u} N^D_{\lambda,\mu} n_{\lambda,\mu,t,u} \binom{u}{s}\binom{i}{t} j^s,$$

where $n_{\lambda,\mu,t,u}$ is the number of path-cycle covers of $D_{\lambda,\mu}$ with t paths and u cycles. Now $\binom{i}{t} j^s$ is a basis for polynomials in two variables, so equating coefficients we see that we just need to prove that for any fixed s and t,

$$\sum_{\lambda,\mu,u} N^D_{\lambda,\mu} n_{\lambda,\mu,t,u} \binom{u}{s} = \sum_{q,r} \delta_D(q, r, s) \binom{q}{t-d+r+q}.$$

We can think of both sides as counting placements of d non-taking rooks on $[d] \times [d]$ with certain multiplicities. On the left-hand side, the number of times each such placement T is counted equals the number of path-cycle covers of drop(T) with exactly t paths plus some number of cycles of which s are distinguished. As for the right-hand side, we can rewrite it as

$$\sum_{e,r} \delta_D(e-r, r, s) \binom{e-r}{t-d+e}.$$

Then for any placement T, only one value of e (namely $e = |\text{drop}(T)|$) involves T. Thus if we let $e = |\text{drop}(T)|$, the number of times T is counted is

$$\sum_r \binom{\text{the number of ways of choosing}}{s \text{ cycles of drop}(T) \text{ with } r \text{ edges}} \cdot \binom{e-r}{t-d+e},$$

which is just the number of ways of choosing s cycles and then deleting $t-(d-e)$ of the remaining edges (i.e., creating $t-(d-e)$ new paths). But $d-e$ is the original number of paths in drop(T), so this results in a total of exactly t paths. The proposition follows. ∎

We remark that Gessel [9] has obtained a generalization of Proposition 10 for the cover polynomials that does not appear to follow from our results.

Our next result generalizes a Möbius inversion formula for factorial polynomials due to Goldman, Joichi and White [16]. For simplicity we consider only the case of acyclic digraphs, although the generalization to arbitrary digraphs is straightforward. So suppose D is an acyclic digraph with d vertices and let B be its associated board. Following an idea of Goldman, Joichi and White, extend the columns of $[d] \times [d]$ infinitely downwards, so that there are now infinitely many rows. Let $\mathscr{S}$ be the set of all placements of d rooks such that

1. every rook lies either on B or one of the appended squares, and
2. no two rooks lie in the same column.

Given $S \in \mathcal{S}$, define $\pi(S)$ to be the partition of $[d]$ in which two numbers i and j lie in the same block if and only if the rooks in columns i and j lie in the same row. To each $S \in \mathcal{S}$ we also associate a coloring of $[d]$ as follows. Color the vertex $i \in [d]$ with color j if the rook in column i lies in the jth *appended* row. Otherwise, if the rook in column i lies in the jth *original* row, make vertex i the same color as vertex j. Since there is exactly one rook in each column, and since D is acyclic, these rules give a well-defined coloring c_S. For every set partition of $[d]$, define

$$T_\pi^D = T_\pi^B \stackrel{\text{def}}{=} \sum_{\{S \in \mathcal{S} \mid \pi(S) = \pi\}} x^S,$$

where

$$x^S \stackrel{\text{def}}{=} \prod_{i \in [d]} x_{c_S(i)}.$$

Finally define

$$T_{\geqslant \pi}^D = T_{\geqslant \pi}^B \stackrel{\text{def}}{=} \sum_{\sigma \geqslant \pi} T_\sigma^B.$$

THEOREM 4. *For any acyclic digraph D with d vertices,*

$$\Xi_D = \sum_{\pi \in \Pi_d} \mu(\hat{0}, \pi)\, T_{\geqslant \pi}^D.$$

Proof. By Möbius inversion, the right-hand side is just $T_{\hat{0}}^D$. The $S \in \mathcal{S}$ such that $\pi(S) = \hat{0}$ are just the placements in which no two rooks lie in the same row or column. The rooks on B then define a path cover and the rooks on the appended rows then ensure that distinct paths are assigned distinct colors. The theorem follows from Proposition 3. ∎

It is not hard to show that this result specializes to [16, Theorem 1(a)]. One might again object that Theorem 4 is contrived because $T_{\geqslant \pi}^D$ is simply a formal device to represent what one gets by Möbius inversion. This time the objection is harder to meet, because $T_{\geqslant \pi}^D$ is not as "nice" an object as $\tilde{\Xi}_{\lambda, \mu}$. For example, type$(\pi)$ = type(σ) does *not* imply $T_{\geqslant \pi}^D = T_{\geqslant \sigma}^D$. Also, Goldman, Joichi and White's well-known factorization theorem [12] for Ferrers shapes, which follows from their Möbius inversion formula, does not seem to have any simple generalization to Ξ_D. Indeed, computing Ξ_D

for various boards appears to be much more difficult than computing factorial polynomials, in part because Ξ_D depends heavily on the embedding of B into $[d] \times [d]$ and not just on the shape of B. However, we do have one result that gives some more information about $T^D_{\geqslant \pi}$.

PROPOSITION 15. *For an acyclic digraph D, the power sum expansion of $T^D_{\geqslant \pi}$ has nonnegative integer coefficients.*

Proof. Let B be the associated board. We have

$$T^D_{\geqslant \pi} = \sum_{\{S \in \mathscr{S} \mid \pi(S) \geqslant \pi\}} x^S.$$

Collect terms that have identical placements of rooks on B. From the definitions we see that each such collection of terms corresponds to the set of colorings of $V(D)$ that are monochromatic on the connected components of the subgraph of D whose edges are those selected by the placement of rooks on B, except that the condition $\pi(S) \geqslant \pi$ imposes the further condition that components which contain elements of the same block of π must always be colored the same color. This gives a power sum symmetric function, so $T^D_{\geqslant \pi}$ is a sum of power sums. ∎

Note that while Theorem 4 resembles Stanley's formula [21, Theorem 2.6]

$$X_G = \sum_{\pi \in L_G} \mu(\hat{0}, \pi)\, p_\pi$$

(where L_G is the lattice of contractions of G), there is a significant difference in that in Theorem 4 all the dependence on the digraph is contained in the $T^D_{\geqslant \pi}$ whereas for X_G all the dependence is contained in L_G. We have obtained variations of Theorem 4 (e.g., by considering rook placements with no two rooks in the same *row*), but so far have found no satisfactory analogue of L_G.

5. EXPANSIONS

It is natural to ask for interpretations of the coefficients when Ξ_D is expanded in terms of various symmetric function bases. In fact, one of the motivations for studying X_G and Ξ_D is a conjecture by Stanley and Stembridge [23, Conjecture 5.5] regarding the elementary symmetric function expansion of X_G. We restate this conjecture here for convenience. Following Stanley [21, Section 5], we write $\mathbf{a} + \mathbf{b}$ for the poset that is a disjoint union of an a-element chain and a b-element chain, and we say that a poset if $(\mathbf{a} + \mathbf{b})$-*free* if it contains no induced subposet isomorphic to $\mathbf{a} + \mathbf{b}$. We also say that a symmetric function g is *u-positive* if $\{u_\lambda\}$ is a symmetric function basis and the expansion of g in terms of this basis has nonnegative coefficients. Then the Stanley–Stembridge conjecture is equivalent to the following.

CONJECTURE 1. *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then $X_{G(P)}$ is e-positive.*

In view of Proposition 2, this conjecture can also be viewed as a conjecture about Ξ_D. One of the most important partial results is the following theorem of Gasharov [8].

THEOREM 5. *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then $X_{G(P)}$ is s-positive.*

We shall prove a slight extension of Gasharov's result that will illustrate the subtlety of Conjecture 1. To state our result we need some definitions.

DEFINITION. A loopless digraph is *weakly $(\mathbf{3}+\mathbf{1})$-free* if, for any ordered pair (u, v) of vertices of D, either D or D' fails to have a directed path of length two from u to v.

Note that weakly $(\mathbf{3}+\mathbf{1})$-free digraphs need not be transitively closed or even acyclic. Our nomenclature is justified by the following proposition.

PROPOSITION 16. *If P is a poset, then P is $(\mathbf{3}+\mathbf{1})$-free if and only if $D(P)$ is weakly $(\mathbf{3}+\mathbf{1})$-free.*

Proof. Saying that $D(P)$ is weakly $(\mathbf{3}+\mathbf{1})$-free is equivalent to saying that if $u \to v \to w$ is a directed path of length two in $D(P)$ and x is any element such that (u, x) is *not* an edge of $D(P)$, then (x, w) is an edge of $D(P)$. Saying that P is $(\mathbf{3}+\mathbf{1})$-free is equivalent to saying that if $u \to v \to w$ is a chain in P and x is any element such that $u \not< x$ in P, then $x < w$ in P. Clearly these two are equivalent. ∎

DEFINITION. Let D be a digraph. A *D-array* is an array

$$\begin{array}{ccc} v_{1,1} & v_{1,2} & \cdots \\ v_{2,1} & v_{2,2} & \cdots \\ \cdots & & \end{array}$$

where each $v_{i,j}$ is either undefined or an element of D and such that

1. for all $i, j \geqslant 1$, if $v_{i,j+1}$ is defined, then $v_{i,j}$ is defined and $(v_{i,j}, v_{i,j+1})$ is an edge of D, and

2. every element of D appears exactly once in the array.

The *shape* of a D-array is the sequence of the lengths of (the defined portion of) the rows. A D-*tableau* is a D-array such that

3. for all $i, j \geqslant 1$, if $v_{i+1,j}$ is defined, then $v_{i,j}$ is defined and $(v_{i+1,j}, v_{i,j})$ is *not* an edge of D.

Our definitions of D-array and D-tableau are motivated by Gasharov's use of Gessel-Viennot [11] P-arrays and P-tableaux in his proof of Theorem 5. We can now state our generalization.

THEOREM 6. *If D is a weakly $(\mathbf{3}+\mathbf{1})$-free digraph, then the coefficient of s_λ in $\Xi_D(x, 0)$ is the number of D-tableaux of shape λ.*

Proof. The proof is almost identical to Gasharov's, and we refer to his paper for some details which we shall omit. Let S_ℓ denote the group of permutations of $[\ell]$. If $\lambda = (\lambda_1, ..., \lambda_\ell)$ is an integer partition and $\pi \in S_\ell$, then we denote by $\pi(\lambda)$ the sequence

$$(\lambda_{\pi(j)} - \pi(j) + j)_{j=1}^{\ell}.$$

Define c_λ by

$$\Xi_D(x, 0) = \sum_\lambda c_\lambda s_\lambda(x).$$

By the same Jacobi–Trudi argument that Gasharov uses,

$$c_\lambda = \sum_{\pi \in S_\ell} (\operatorname{sgn} \pi) \cdot \left(\text{coefficient of } \prod x_i^{\pi(\lambda)_i} \text{ in } \Xi_D(x, 0) \right),$$

where $\operatorname{sgn} \pi$ is the sign of the permutation π. Now by Proposition 3, $\Xi_D(x, 0)$ counts path colorings of D, and path colorings of D are in bijection with D-arrays (the rows of the D-array give the directed paths and the path in row i is assigned the color i). If we let

$$A = \{(\pi, T) \mid \pi \in S_\ell \text{ and } T \text{ is a } D\text{-array of shape } \pi(\lambda)\},$$

it then follows that

$$c_\lambda = \sum_{(\pi, T) \in A} \operatorname{sgn} \pi.$$

Now let

$$B = \{(\pi, T) \in A \mid T \text{ is } \textit{not} \text{ a } D\text{-tableau}\}$$

and note that if T is a D-tableau, then $\pi(\lambda)_1 \geqslant \pi(\lambda)_2 \geqslant \cdots$ so that π must be the identity permutation. Thus to prove the theorem it suffices to find an involution $\varphi\colon B \to B$ such that if $(\sigma, T') = \varphi(\pi, T)$ then $\operatorname{sgn} \sigma = -\operatorname{sgn} \pi$. Gasharov's involution works without modification; for completeness we restate it here. If

$$T = \begin{matrix} v_{1,1} & v_{1,2} & \cdots \\ v_{2,1} & v_{2,2} & \cdots \\ \cdots & & \end{matrix}$$

then let $c = c(T)$ be the smallest positive integer such that condition 3 fails for $j = c$ and some i. Let $r = r(T)$ be the largest i with this property. Define $\sigma = \pi \circ (r, r+1)$ where $(r, r+1)$ is the permutation that interchanges r and $r+1$. Define

$$T' = \begin{matrix} u_{1,1} & u_{1,2} & \cdots \\ u_{2,1} & u_{2,2} & \cdots \\ \cdots & & \end{matrix}$$

by letting

(a) $u_{i,j} = v_{i,j}$ if $i \neq r$ or $i \neq r+1$ or $(i = r$ and $j \leqslant c-1)$ or $(i = r+1$ and $j \leqslant c)$;

(b) $u_{r,j} = v_{r+1,j+1}$ if $j \geqslant c$ and $v_{r+1,j+1}$ is defined;

(c) $u_{r+1,j} = v_{r,j-1}$ if $j \geqslant c+1$ and $v_{r,j-1}$ is defined.

(Other values of the array T' remain undefined.) Now row $r+1$ of T' satisfies condition 1, because if $v_{r,c}$ is defined then $(v_{r+1,c}, v_{r,c})$ is an edge of D by definition of r and c. To show that T' is a D-array it suffices to show that row r satisfies condition 1 since condition 2 is obviously satisfied. Possible trouble arises only if $c \geqslant 2$, but then $v_{r+1,c-1} \to v_{r+1,c} \to v_{r+1,c+1}$ is a path of length two in D and $(v_{r+1,c-1}, v_{r,c-1})$ is *not* an edge in D, so it follows from the assumption that D is weakly $(\mathbf{3}+\mathbf{1})$-free that $(v_{r,c-1}, v_{r+1,c+1})$ is an edge of D, and condition 1 is met. Now $(v_{r+1,c}, v_{r+1,c+1})$ is an edge of D so if $u_{r,c}$ is defined $(u_{r+1,c}, u_{r,c})$ is an edge of D (since $u_{r+1,c} = v_{r+1,c}$ and $u_{r,c} = v_{r+1,c+1}$) and thus T' is not a D-tableau. It is clear that T' has shape $\sigma(\lambda)$ and that $c(T') = c(T)$ and $r(T') = r(T)$. Also, $\operatorname{sgn} \sigma = -\operatorname{sgn} \pi$, so φ is the desired sign-reversing involution. ∎

Note that a digraph is weakly $(\mathbf{3}+\mathbf{1})$-free if and only if its complement is weakly $(\mathbf{3}+\mathbf{1})$-free, so Corollary 2 applied to Theorem 6 does not enlarge the class of known s-positive path-cycle symmetric functions.

It is natural to conjecture that if D is weakly $(\mathbf{3}+\mathbf{1})$-free then $\Xi_D(x, 0)$ is e-positive but for instance if we let D be the digraph with adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we find (with the aid of John Stembridge's SF package for Maple) that

$$\Xi_D(x, 0) = s_4 + 2s_{31} + s_{22} + 4s_{211} + 3s_{1111} = 3e_{31} - e_{211} + e_{1111}.$$

In fact, of the five essentially distinct weakly $(\mathbf{3}+\mathbf{1})$-free acyclic digraphs on four vertices that are not transitively closed, only one is e-positive. So the way the property of being $(\mathbf{3}+\mathbf{1})$-free is used in Gasharov's proof is far from enough to yield e-positivity even if the condition of acyclicity is added. This shows how delicate Conjecture 1 is.

We conclude this section with a theorem that is closely related to the result of Stanley [21, Corollary 2.7] that ωX_G is p-positive for all graphs G. Corollary 2 shows that a direct analogue is not possible, but we do have the following result.

THEOREM 7. *If D is an acyclic digraph, then $\omega_x \Xi_D$ is p-positive.*

Proof. Since D is acyclic, all path-cycle covers are path covers, and $\Xi_D = \Xi_D(x, 0)$. From Doubilet [6, Appendix 1] we know that for any set partition π,

$$\tilde{m}_\pi = \sum_{\sigma \geqslant \pi} \mu(\pi, \sigma)\, p_\sigma.$$

Thus

$$\begin{aligned} \Xi_D &= \sum_S \sum_{\sigma \geqslant \pi(S)} \mu(\pi(S), \sigma)\, p_\sigma \\ &= \sum_\sigma \left(\sum_{\{S \mid \pi(S) \leqslant \sigma\}} \mu(\pi(S), \sigma) \right) p_\sigma, \end{aligned}$$

where S ranges over path covers. Now fix σ and let $D_1, D_2, ..., D_l$ be the subgraphs induced by the blocks of σ, with sizes $d_1, d_2, ..., d_l$ respectively. If c_i is the coefficient of p_{d_i} in Ξ_{D_i}, then we claim that the coefficient of p_σ in Ξ_D is $\prod_i c_i$. To see this, first note that choosing a path cover S of D such that $\pi(S) \leqslant \sigma$ is equivalent to (independently) choosing path covers for each D_i. Now it is well known and easy to see that the interval $[\hat{0}, \sigma]$ in a partition lattice is isomorphic to

$$\Pi_{d_1} \times \Pi_{d_2} \times \cdots \times \Pi_{d_l}.$$

It is also well known (e.g., [22, Prop. 3.8.2]) that the Möbius function of a product is the product of the Möbius functions. Putting these facts together readily yields our claim.

Thus to prove the theorem it suffices to prove that for an acyclic digraph with d vertices the sign of the coefficient of p_d is $(-1)^{d-1}$. For then, since any induced subgraph of an acyclic graph is acyclic, we can apply our claim above to show that the coefficient of p_σ is $\operatorname{sgn} \sigma$.

Let D have d vertices. By specializing via Proposition 8, we see that the coefficient of p_d in Ξ_D equals the coefficient of i in $R(D; i)$. Directly from the definitions we see that this equals

$$(-1)^{d-1} \sum_{k=0}^{d-1} (-1)^k \, r_k^D (d-k-1)!$$

Now an acyclic digraph as at least one source and one sink, so by removing the corresponding row and column we see that B may be regarded as a subset of a $(d-1) \times (d-1)$ board. Then the above sum is a positive integer, by the basic inclusion-exclusion formula for rooks (see [22, Theorem 2.3.1]). The sign of the coefficient is therefore $(-1)^{d-1}$ as desired. ∎

We mention in passing that the argument used in the above proof allows [21, Proposition 5.5] to be extended from posets to acyclic digraphs.

6. Future Work

It is clear that we have only scratched the surface in our investigations in this paper. We list a few promising directions for further research.

1. The involution ω plays an important role in the theory of symmetric functions. Does the involution

$$g(x, y) \mapsto [\omega_x g(x, -y)]_{x \to (x, y)}$$

play an important role in the theory of symmetric functions in two sets of variables? It is not even immediately obvious that this *is* an involution, so its properties could be quite subtle. In this regard, we state without proof that if we define

$$\hat{\Xi}_D(x, y) \stackrel{\text{def}}{=} \sum_S (-2)^{\ell(\sigma(S))} \, \tilde{m}_{\pi(S)}(x, y) \, p_{\sigma(S)}(y),$$

where the sum is over all path-cycle covers S of D, then

$$\hat{\Xi}_{D'}(x, y) = \omega_x \hat{\Xi}_D(x, -y),$$

and this transformation *is* clearly an involution, so perhaps $\hat{\Xi}_D$ should be studied alongside Ξ_D.

2. Can any more rook theory be generalized to Ξ_D? For example, can we determine which boards have equal path-cycle symmetric functions? Or can we compute the path-cycle symmetric function of some simple boards? It is not difficult to show that if D is a directed path with d vertices then

$$\Xi_D = \sum_{\lambda \vdash d} \ell(\lambda)!\, m_\lambda = \sum_{r=0}^{d-1} u_r s_{d-r, 1^r}$$

where u_r is the number of permutations of $r+1$ with no consecutive ascending pairs. A similar result holds for directed cycles, but we have not been able to compute Ξ_D for any other significant class of digraphs.

3. What more can be said about the symmetric function bases $\tilde{\Xi}_\lambda$ and $\tilde{\Xi}_{\lambda, \mu}$? Can they be fitted into the Möbius inversion framework of Doubilet [6]?

4. Is there a natural representation of the symmetric group corresponding to Ξ_D for weakly $(\mathbf{3}+\mathbf{1})$-free digraphs? If such a representation could be constructed it might shed some light on Conjecture 1.

Acknowledgments

I am grateful to Richard Stanley, my thesis advisor, for drawing my attention to this problem and for his numerous helpful suggestions and comments. I also thank Ira Gessel for permitting me to reproduce his proof of Corollary 2. John Stembridge's SF package for Maple was used extensively in the research leading to this paper. This work was supported in part by a National Science Foundation graduate fellowship.

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