Perfect Matching Conjectures and Their Relationship to P $\neq$ NP

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Distance Degree Regular Graphs

- **If *v* is a vertex in a graph *G*, let $d_i(v)$ be the number of vertices in *G* at a distance *i* from *v***
- ***G* is distance degree regular (DDR) if for all *i*, $d_i(v)$ depends only on *i* and not on *v***
- **Examples: vertex-transitive graphs, strongly regular graphs, distance-regular graphs**
- **Conjecture 1: Let *G* be DDR; then *G* has a perfect matching iff every connected component is even**

Gallai-Edmonds Decomposition

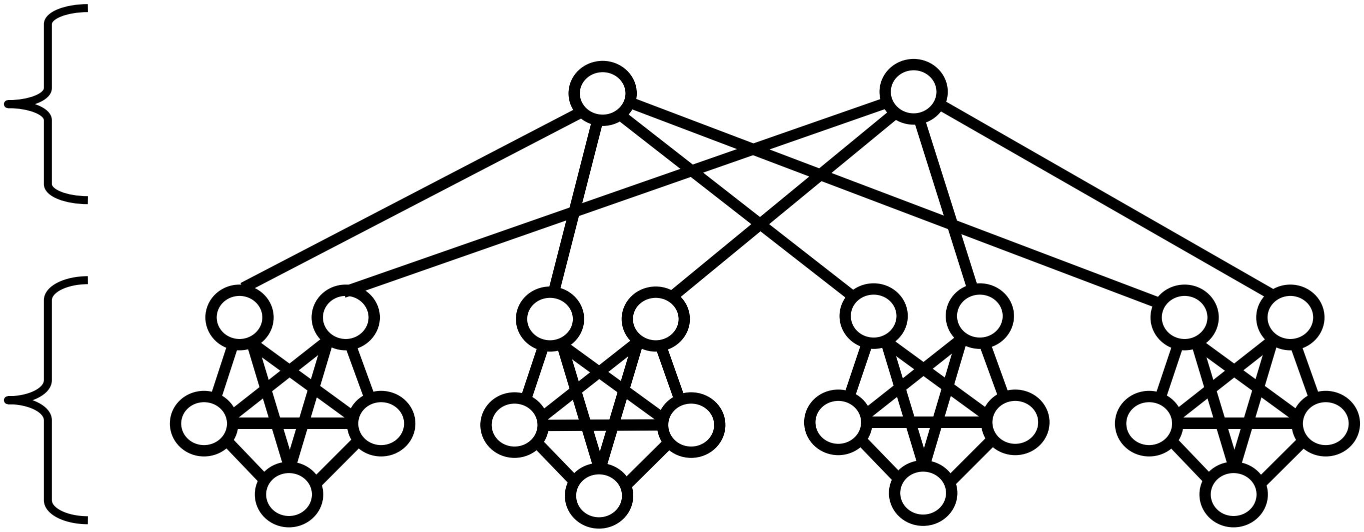
- **Given any graph G, define:**
  - **$D(G)$ = { v : $\exists$ maximum matching of G not containing v }**
  - **$A(G)$ = neighbors of $D(G)$ not already in $D(G)$**
  - **$C(G)$ = the remaining vertices of G**
- **Then:**
  - **G has a perfect matching iff $D(G) = \emptyset$**
  - **Every component of $D(G)$ is odd**
  - **If G has no perfect matching then $A(G)$ is a Tutte set, i.e., it has fewer vertices than $G - A(G)$ has odd components**
  - **$C(G)$ has a perfect matching**
  - **[Other conclusions omitted]**

Gallai-Edmonds Example

$C(G) = \varnothing$

$A(G)$

$D(G)$



Partial Results on Conjecture 1

- **Recall Conjecture 1: If *G* is DDR then *G* has a perfect matching iff every component is even**

- **True for vertex-transitive graphs**
  - **Exercise in Lovász and Plummer (follows easily from Gallai-Edmonds decomposition)**

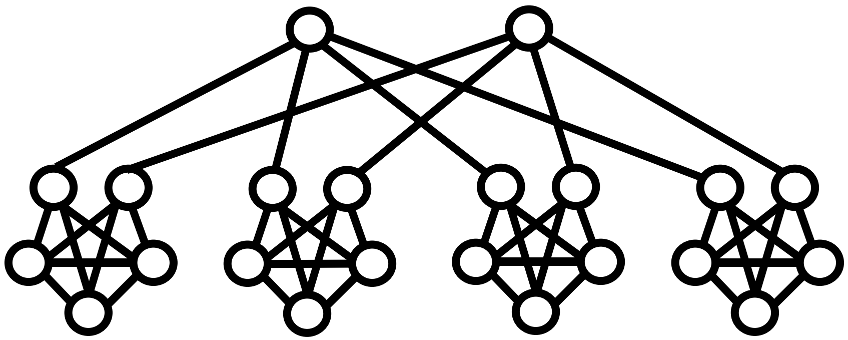
- **True for strongly regular graphs (Holton and Lou)**

- **True for all DDR graphs of diameter 2 or 3**

- **True for DDR graphs of diameter 4 satisfying certain additional technical conditions**

Strengthenings of Conjecture 1

- **"DDR" cannot be strengthened to "self-centered"**
  - **A graph is self-centered if the largest *i* such that $d_i(v) \neq 0$ is independent of *v***



- **Conjecture 1´: If G is a connected graph, $u \in D(G)$ and v $\notin D(G)$ then $d_i(u) \neq d_i(v)$ for some *i***

Multiregular Multipartite Graphs

- **Let *G* be a graph equipped with a multiregular multipartition $V(G) = V_1 \cup \ldots \cup V_k$**
  - **If $v \in V_i$ then the number r_{ij} of neighbors of v in V_j depends only on i and j and not on v, and $r_{ii} = 0$**

- **Conjecture 2: If *G* has no perfect matching, then *G* has a Tutte set that is a union of V_i's**

- **True for the multipartition into single vertices and the multipartition into the vertex orbits of the automorphism group of *G***

Listing Polytime Graph Properties

- **There is a computable function *M*(*n*) such that**
  - **For all *n*, *M*(*n*) is a polytime Turing machine that recognizes a property *X* of labeled graphs**
  - **For every polytime property *X* of labeled graphs, there exists *n* such that *M*(*n*) recognizes *X***
- **One simply enumerates all polynomially clocked Turing machines**
- **But what if we restrict ourselves to polytime isomorphism-invariant properties of graphs (i.e., polytime properties of unlabeled graphs)?**

Central Problem of Finite Model Theory

- **Is there a computable function *M**(*n*) such that**
  - **For all *n*, *M**(*n*) is a polytime Turing machine that recognizes a property *X* of unlabeled graphs**
  - **For every polytime property *X* of unlabeled graphs, there exists *n* such that *M**(*n*) recognizes *X* ?**

- **Fact: If not, then P $\neq$ NP**
  - **Graph canonization is in $P^{NP} = \Delta_2$, the second level of the polynomial hierarchy**
  - **If P = NP then graph canonization is in P, so each unlabeled graph can be replaced with a canonical labeled representative**

Why Is This "Finite Model Theory"?

- **Candidates for *M**(*n*) are typically defined by creating a formal language (or logic) with**
  - **computable syntax, i.e., a computable set of formulas**
  - **computable semantics, i.e., a computable correspondence between formulas and Turing machines (for the polytime graph properties expressed by the formulas)**
- **Hence the central problem is often stated: Is there a logic that [strongly, effectively] captures P on unordered structures?**
- **Best-known example of a logic: First-order logic**
  - **Much too weak to express all polytime properties**

Two Logics for Unordered Structures

- **FP+C (fixed-point logic with counting)**
  - **FP+C, in fact FP, captures P on ordered structures (Immerman 1982, Vardi 1982)**
- **CPT+C (choiceless polytime with counting)**
  - **"Choiceless" means no non-canonical choices allowed**

- **FP+C $\subset$ P (Cai-Fürer-Immerman 1989)**
- **FP+C $\subset$ CPT+C $\subseteq$ P (Blass-Gurevich-Shelah 2002)**

- **It is open whether FP+C or CPT+C can capture perfect matchability of unlabeled graphs**
  - **Leads to the conjectures in the first part of the talk**

- **Resolving the central problem either way would be a major result, so any partial results in either direction are interesting**

- **These investigations often boil down to concrete combinatorial problems that require no special knowledge of logic**

- **More combinatorialists are needed in this field!**

Selected References

- A. Blass, Y. Gurevich and S. Shelah, "On polynomial time computation over unordered structures," *J. Symbolic Logic* **67** (2002), 1093–1125 [most important reference]
- J.-Y. Cai, M. Fürer and N. Immerman, "An optimal lower bound on the number of variables for graph identification," *Proc. 30th IEEE FOCS* (1989), 612–617
- H.-D. Ebbinghaus and J. Flum, *Finite Model Theory*, 2nd ed., Springer-Verlag, Berlin, 1999
- D. A. Holton and D. J. Lou, "Matching extensions of strongly regular graphs," *Australas. J. Combin.* **6** (1992), 187–208
- N. Immerman, "Relational queries computable in polynomial time," *Proc. 14th ACM STOC* (1982), 147–152
- L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam, 1986
- M. Otto, *Bounded Variable Logics and Counting: A Study in Finite Models*, Springer-Verlag, Berlin, 1997
- M. Y. Vardi, "Complexity of relational query languages," *Proc. 14th ACM STOC* (1982), 137–146