We are given a Hessenberg function $\mathbf{m}$. Let P denote the associated natural unit interval order and let G denote the incomparability graph of P. See Shareshian–Wachs for the definition of a *P-tableau* and of a *G-inversion* of a P-tableau.

Lemma 1. If T is a P-tableau with a unique G-inversion, then the G-inversion has the form $\{i, i+1\}$ for some i in the range $1 \le i \le n-1$.

Proof. Suppose that $\{j, k\}$ is a G-inversion with $1 \le j < k \le n$. Then by definition, j appears in a lower row of T than k does, and there is an edge between j and k in G. It follows that there exists i with $j \le i < k$ such that i appears in a lower row of T than $i+1$ does. By the definition of a unit interval order, $\{i, i+1\}$ must be an edge of G. Therefore, $\{i, i+1\}$ is a G-inversion. We have shown that if there is any G-inversion at all, then there must be a G-inversion of the form $\{i, i+1\}$, so if the G-inversion is unique then it must have this form. □

Lemma 2. If T is a P-tableau and $\{j, k\}$ with $j < k$ is *not* a G-inversion of T, then k does *not* appear immediately above j (i.e., in the same column and in the preceding row).

Proof. If j and k are adjacent in G then k appearing immediately above j would constitute a G-inversion, which we have ruled out by hypothesis. On the other hand, if j and k are *not* adjacent in G, then because P is a *natural* unit interval order, we must have $j \prec k$ in P, and k appearing immediately above j would violate the definition of a P-tableau. □

Assumption. From now on, assume that $m_i \ge i+1$ for all $i < n$ (i.e., that G is connected).

Define a_i by setting $a_0 = 0$ and, for $1 \le i \le n-2$, by setting

$$a_i = \begin{cases} 0, & \text{if } m_i \ge i+2; \\ 1, & \text{if } m_i = i+1 \text{ and either } m_{i+1} \ge i+3 \text{ or } i = n-2; \\ i+1, & \text{otherwise.} \end{cases}$$

Theorem. For $1 \le j \le n-1$, let $\mathcal{T}_j$ be the set of P-tableaux whose unique G-inversion is $\{j, j+1\}$. Then for $0 \le i \le n-2$,

$$e_{a_i} e_{n-a_i} = \sum_{T \in \mathcal{T}_{i+1}} s_{\text{shape}(T)}.$$

Before proving the Theorem, let us note a simple corollary.

Corollary 1. The coefficient of t in $X_G(t)$ is $\sum_{i=0}^{n-2} e_{a_i} e_{n-a_i}$.

Proof. Theorem 6.3 of Shareshian–Wachs says that the coefficient of t in $X_G(t)$ is $\sum_T s_{\text{shape}(T)}$, where the sum is over all P-tableaux with a single G-inversion. By Lemma 1, every P-tableau belongs to exactly one $\mathcal{T}_i$. So Corollary 1 follows from the Theorem. $\square$

The proof of the Theorem requires some preliminaries.

Lemma 3. Let T be a P-tableau. If $1 \le i < n$ and $\{i, i+1\}$ is *not* a G-inversion, then $i+1$ appears in a lower row of T than i does.

Proof. By the Assumption, i and $i+1$ are adjacent in G. So by the definition of a G-inversion, $i+1$ cannot appear in a higher row than i. Moreover, i and $i+1$ cannot appear in the same row either, because by the definition of a P-tableau, elements in the same row must form a totally ordered subset of P and hence cannot be adjacent in G. $\square$

Proposition 1. Let T be a P-tableau with exactly one G-inversion $\{i, i+1\}$, and let r_j denote the row in which j appears. Then T has at most two columns. If T has a single column then the entries down that column are

$$1, 2, 3, \ldots, i-2, i-1, i+1, i, i+2, i+3, \ldots, n.$$

If T has two columns then the first i entries in column 1 are the numbers from 1 to i in order, the first entry in column 2 is $i+1$, and the sequence $r_{i+2}, r_{i+3}, \ldots, r_n$ is strictly increasing.

Proof. By Lemma 3, the sequence $r_1, r_2, \ldots, r_i$ is strictly increasing, and the sequence $r_{i+1}, r_{i+2}, \ldots, r_n$ is strictly increasing. Therefore the only entries that can possibly appear in the first row of T are 1 and $i+1$, so T has at most two columns.

If T has a single column, then by Lemma 2, the entries down column 1 must be arranged in increasing numerical order except that $i+1$ can appear before i (and in fact *must* appear before i since $\{i, i+1\}$ is a G-inversion). The only possibility is therefore the one stated in Proposition 1.

If T has two columns then as noted above, the entries in row 1 are 1 and $i+1$, with 1 appearing in column 1 and $i+1$ appearing in column 2 since the entries in a row of a P-tableau must appear in increasing order according to the partial order P, and P is a natural unit interval order. The remaining entries in column 2 cannot contain any number from 1 to $i-1$ or else there would be a violation of Lemma 2 in column 2, so the numbers 1 to $i-1$ must appear in column 1; moreover, they must be the *first* $i-1$ entries in column 1 or else there would be a violation of Lemma 2 in column 1.

We can now narrow down the possibilities for the position of i in T to either row 2, column 2, or row i, column 1, because otherwise something larger than $i+1$ would appear directly above i, violating Lemma 2. But row 2, column 2 is not actually possible, because $\{i-1, i\}$ is not a G-inversion, so by Lemma 3, $i-1$ would have to be in row 1, forcing

$i-1=1$, i.e., $i=2$, but then whatever is in row 2, column 1 will be larger than 2, violating the definition of a P-tableau in row 2. We have now proved all the assertions of Proposition 1. $\square$

Proof of Theorem. Recall the well-known expansion

$$e_\mu = \sum_{\lambda:\lambda'\geq\mu} K_{\lambda',\mu} s_\lambda$$

where the coefficients are the Kostka numbers and $\geq$ denotes the dominance order. The Theorem concerns only the case in which μ has at most two parts, and in this case, $K_{\lambda',\mu}=1$ if $\lambda'\geq\mu$ and $K_{\lambda',\mu}=0$ otherwise. By Proposition 1, we know that the P-tableaux in $\mathcal{T}_{i+1}$ have at most two columns. So to prove the Theorem, it suffices to prove that for each j in the range $0\leq j\leq \min(a_i, n-a_i)$, there is a unique P-tableau in $\mathcal{T}_{i+1}$ whose second column has height j.

First let us show that there is always exactly one single-column P-tableau in $\mathcal{T}_{i+1}$. Proposition 1 tells us that there is at most one single-column P-tableau, and says what it must be if it exists. The only thing to note is that this is indeed a P-tableau, because by the Assumption, $i+1$ is adjacent to $i+2$, so the appearance of $i+2$ directly above $i+1$ does not violate the definition of a P-tableau.

Now observe that if $i=0$ then there cannot be a two-column $T\in\mathcal{T}_{i+1}=\mathcal{T}_1$ because T would have to have 1 and 2 in row 1, which is not possible because the Assumption implies that 1 and 2 are incomparable in P. So $\mathcal{T}_1$ contains only the single-column P-tableau. Since $a_0=0$, this proves the Theorem in this case. From now on, let us assume that $i\geq 1$.

If $m_i\geq i+2$, then i and $i+2$ are adjacent in G. We claim that $T\in\mathcal{T}_{i+1}$ cannot have two columns. If it did, then by Proposition 1, the entry in row 1, column 2 would be $i+2$, and i would appear somewhere in column 1. Since i and $i+2$ are adjacent, they cannot appear in the same row of T, so i would appear in a lower row than $i+2$ did, and $\{i, i+2\}$ would be a G-inversion, which is not allowed by Lemma 1. Therefore $\mathcal{T}_{i+1}$ contains only the single-column P-tableau, and since $a_i=0$ when $m_i\geq i+2$, the Theorem is proved in this case as well.

Next, suppose that $m_i=i+1$ and either $m_{i+1}\geq i+3$ or $i=n-2$. We claim that the only $T\in\mathcal{T}_{i+1}$ with two columns has a single element in column 2, namely $i+2$ in row 1. By Proposition 1, any other two-column P-tableau would have $i+3$ in row 2, column 2. If $i=n-2$ then this is impossible simply because $i+3>n$. Otherwise, the argument is similar to the one in the previous paragraph: $i+1$ appears in column 1, row $i+1\geq 2$; the fact that $m_{i+1}\geq i+3$ implies that $i+1$ and $i+3$ are adjacent, so they cannot appear in the same row, so $i+1$ appears in a *lower* row than $i+3$, so $\{i+1, i+3\}$ is a G-inversion, contradicting Lemma 1. So we just need to verify that T, which has $i+2$ in row 1, column 2, and all the remaining entries in increasing order down column 1, is indeed a P-tableau with unique G-inversion $\{i+1, i+2\}$. Certainly T has the G-inversion

$\{i+1, i+2\}$ because $i+1$ appears in row $i+1 > 1$ (which is lower than row 1, where $i+2$ appears), and $i+1$ and $i+2$ are adjacent by the Assumption. Also, since $m_i = i+1$, it follows that $i+2 \succ j$ in P for all $j < i$. That means that there are no other G-inversions in T, and also the two elements 1 and $i+2$ in row 1 do form a chain in P. Since $a_i = 1$, this proves the Theorem in this case as well.

Finally, we are left with the case $m_i = i+1$, $m_{i+1} = i+2$, and $1 \leq i < n-2$. The P-tableaux with two columns allowed by Proposition 1 are obtained by picking j in the range $1 \leq j \leq \min(a_i, n-a_i)$, then placing the first j numbers in the set $\{i+2, i+3, \ldots, n\}$ in increasing order down column 2, and placing the remaining numbers in that set in increasing order down column 2 (underneath $i+1$). So we just need to show that all these are actually valid P-tableaux. The argument in the previous paragraph shows that since $m_i = i+1$, there are no illegitimate G-inversions where $i+2$ is the larger number, and row 1 is a chain in P. The fact that $m_{i+1} = i+2$ similarly shows that if $j \leq i+1$ and $k \geq i+3$ then $j \prec k$ in P so all the rows of T are indeed chains in P and there are no illegitimate G-inversions where the larger number is greater than $i+2$. This completes the proof. $\square$