The purpose of this note is to give a combinatorial proof that for any indifference graph G, the set

$$P_G := \{\lambda : \langle X_G, s_\lambda \rangle \neq 0\}$$

of partitions λ such that the coefficient of s_λ in the Schur-function expansion of X_G is nonzero has a unique maximal element μ in dominance order.

For the proof, it is useful to think of the vertices of G as represented explicitly by a finite set of closed unit intervals in the real line. From now we use the terms "vertex" and "unit interval" interchangeably. Let v_i denote the ith unit interval, counting from the left. The following algorithm is crucial.

First Fit Coloring Algorithm. Assign a color (i.e., a positive integer) to each v_i as follows. Go through the vertices in the order v_1, v_2, v_3, etc., and at each step, assign to the current v_i the smallest available color, i.e., the smallest color that has not already been assigned to some v_j that overlaps with v_i. Let f denote the coloring so obtained; i.e., for every vertex v, let $f(v)$ be the color assigned to v by the First Fit Coloring Algorithm.

Example. If G is a path, then the First Fit Coloring Algorithm will assign colors $1, 2, 1, 2, 1, \ldots$ to the vertices.

Definition. Given a coloring c of G, let shape(c) be the sequence whose kth element is the number of vertices $v \in G$ such that $c(v) = k$.

Example. If G is a path with n vertices, then shape(f) is the sequence $\lceil n/2 \rceil, \lfloor n/2 \rfloor$.

Theorem 1. shape(f) is the unique maximum element of P_G in dominance ordering.

At the moment it is not even obvious that shape(f) is a weakly decreasing sequence, but this will follow as a corollary of the following theorem.

Theorem 2. Let c be any coloring of G. Then shape(f) $\geq$ shape(c).

Proof. Let u be the leftmost unit interval such that $c(u) \neq f(u)$. We now construct another coloring c' such that

1. $c'(v) = f(v)$ for all v weakly to the left of u, including u itself, and

2. shape(c') $\geq$ shape(c).

This will prove the theorem, because iterating the procedure will eventually produce f, and at each step, the shape will increase in dominance order.

Note first that $c(u) > f(u)$ since if $c(u)$ were less than $f(u)$ then the First Fit Coloring Algorithm would have assigned the smaller color $c(u)$ to u. We tentatively construct $c'(u)$ by setting $c'(v) = c(v)$ for all v except $v = u$ and to set $c'(u) = f(u)$. If this results in a legal coloring c' then we stop. Otherwise, u overlaps with some v further to the right such that $c(v) = f(u)$. The geometry of unit intervals implies that there is at most one such v (if there were more than one then they would themselves overlap and c could not assign them both the same color) so if this happens, we set $c'(v) = c(u)$. Again, we stop if c' is now a legal coloring. If c' is not legal, then the only possible problem is that there is another unit interval w overlapping with v to the right of v such that $c(w) = c(u)$, and if this happens then we set $c'(w) = f(u)$. We continue the process in the obvious way; it must terminate because we are moving further and further to the right at each step. The end result is that c' agrees with c except that the colors of some unit intervals have switched from $c(u)$ to $f(u)$ or vice versa. The number that switch from $c(u)$ to $f(u)$ is either equal to or one more than the number that switch from $f(u)$ to $c(u)$, and since $c(u) > f(u)$, it follows that $\mathrm{shape}(c') \geq \mathrm{shape}(c)$. This completes the proof.

The property asserted by Theorem 2, that there exists a coloring of the graph whose shape dominates that of every other coloring, is something that I'm sure has come up in the literature before, but it seems surprisingly hard to track down references. Years ago, I formulated a conjecture that a certain class of graphs has this property, and that conjecture is still open to this day. I remember that when I first formulated that conjecture, I similarly had a hard time tracking down references.

Anyway, given Theorem 2, it follows that $\mathrm{shape}(f)$ is weakly decreasing, because otherwise we could permute the colors to obtain another coloring f' with $\mathrm{shape}(f')$ weakly decreasing and $\mathrm{shape}(f')$ would strictly dominate $\mathrm{shape}(f)$.

This means that if we construct a P-array by taking the vertices of G one at a time from left to right and appending it to the end of the first row that we can, then at each stage we maintain the shape of a Young diagram (i.e., the row lengths weakly decrease), because we are just applying the First Fit Coloring Algorithm to the subgraph formed by an initial segment of the unit intervals.

Finally, we note that the resulting P-array is actually a P-tableau. A violation of the column constraint would mean u directly below v with the unit interval u being to the left of the unit interval v, but this is not possible because v must be placed earlier than u in the First Fit Coloring Algorithm, and therefore v must lie weakly to the left of u.

Thus there exists a P-tableau with shape equal to $\mathrm{shape}(f)$. The shape of any other P-tableau is equal to $\mathrm{shape}(c)$ for some coloring c and is therefore dominated by $\mathrm{shape}(f)$. This proves Theorem 1.