A NEW CHARACTERIZATION OF THE FIBONACCI-FREE PARTITION

Timothy Chow
226 Shady Avenue, Apt. 308, Pittsburgh, PA 15206
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1. Introduction

There exists a unique partition of the positive integers into two disjoint sets A and B such that no two distinct integers from the same set sum to a Fibonacci number (see [1], [2], and [3]). For the purposes of this paper, we shall refer to this partition as the "Fibonacci-free partition." The first few numbers in the sets A and B are:

$$A = \{1, 3, 6, 8, 9, 11, \ldots\},$$

$$B = \{2, 4, 5, 7, 10, 12, \ldots\}.$$

In this paper, we shall prove that the sets A and B can be written in the form

$$A = \{[n\phi]\} - \{[m\phi] \mid \text{fp}(m\phi) > \phi/2\},$$

$$B = \{[n\phi^2]\} \cup \{[m\phi] \mid \text{fp}(m\phi) > \phi/2\},$$

where m is a positive integer, $\phi = (1 + \sqrt{5})/2$, n ranges over all the positive integers, and $\text{fp}(x)$ denotes the fractional part of x. (We depart from the standard notation where (x) denotes the fractional part of x to avoid confusion in complicated expressions. See Lemma 4.4 below, for instance.) We shall also prove the following conjecture of Chris Long [4]: the set A satisfies the equality $A = \{[n\phi]\} - A'$, where $A' = \{[s\phi^3] \mid s \in A\}$.

We remark that in [3] it is shown that the Fibonacci-free partition cannot be expressed in the form $A = \{[na]\}$, $B = \{[nb]\}$ for any a and b, but that the above result shows that such a representation is "almost" possible.

A note on notation: in this paper, unless otherwise specified, F_n denotes the n^{th} Fibonacci number, $\lceil x \rceil$ denotes the least integer $\geq x$, and $\text{dist}(x)$ is the distance of x from the nearest integer, i.e.,

$$\text{dist}(x) = \min\{x - [x], \lceil x \rceil - x\}.$$

2. An Important Lemma

Definition: A positive integer a is said to have the *distance property* if

$$\text{dist}(a\phi) > \text{dist}(\phi F)$$

for all Fibonacci numbers $F > a$.

Lemma 2.1: All positive integers have the distance property.

This crucial lemma is the key to the proof of Theorem 3.4 below. It will be used in the proofs of all three lemmas in the next section.

Proof: We proceed by induction. Note first of all that 1 has the distance property. So now suppose that there exists $F_n \geq 2$ such that all integers $\leq F_n$ have the distance property. We have to show that all integers $\leq F_{n+1}$ also have the distance property. It is well known that Fibonacci numbers have the distance property. So we need only check that if k is any integer such that $F_n < k < F_{n+1}$, then $\text{dist}(k\phi) > \text{dist}(\phi F_{n+1})$.

The case $k = F_{n+1}$ is clear; therefore, we can safely assume $k < F_{n+1}$. Let $m = k - F_n$. Then m is a positive integer $< F_{n-1}$ so that $\text{dist}(m\phi) > \text{dist}(\phi F_{n-1})$, by the induction hypothesis. There are two cases to consider:

(1) $\phi F_{n+1} > F_{n+2}$. Then $\text{dist}(\phi F_{n+1}) = \text{fp}(\phi F_{n+1})$. So to show that $\text{dist}(\phi F_{n+1}) < \text{dist}(k\phi)$, we just need to show two things:

$$\text{fp}(\phi F_{n+1}) < \text{fp}(k\phi) \quad \text{and} \quad \text{fp}(\phi F_{n+1}) < 1 - \text{fp}(k\phi).$$

First of all, note that $\text{fp}(m\phi) > \text{fp}(\phi F_{n-1})$, since

$$\text{fp}(m\phi) \geq \text{dist}(m\phi) > \text{dist}(\phi F_{n-1}) = \text{fp}(\phi F_{n-1}).$$

Now, $\text{dist}(\phi F_n) < \text{dist}(\phi F_{n-1})$, and since $\text{dist}(\phi F_n) = F_{n+1} - \phi F_n$, this means that $\text{fp}(\phi F_{n-1}) - (F_{n+1} - \phi F_n) > 0$. Thus,

$$\begin{aligned}\text{fp}(\phi F_n) + \text{fp}(\phi F_{n-1}) &= \phi F_n - (F_{n+1} - 1) + \text{fp}(\phi F_{n-1})\\ &= 1 + \text{fp}(\phi F_{n-1}) - (F_{n+1} - \phi F_n)\\ &> 1.\end{aligned}$$

But $\text{fp}(m\phi) > \text{fp}(\phi F_{n-1})$, so $\text{fp}(\phi F_n) + \text{fp}(m\phi) > 1$. By the definition of m above, $k\phi = \phi F_n + m\phi$. It follows that

$$\begin{aligned}\text{fp}(k\phi) &= \text{fp}(\phi F_n) + \text{fp}(m\phi) - 1\\ &> \text{fp}(\phi F_n) + \text{fp}(\phi F_{n-1}) - 1\\ &= \text{fp}(\phi F_{n+1}).\end{aligned}$$

It remains to be shown that $\text{fp}(\phi F_{n+1}) < 1 - \text{fp}(k\phi)$. We have

$$\begin{aligned}\text{fp}(k\phi) &= \text{fp}(\phi F_n) + \text{fp}(m\phi) - 1\\ &< \text{fp}(\phi F_n)\\ &= 1 - \text{dist}(\phi F_n)\\ &< 1 - \text{dist}(\phi F_{n+1})\\ &= 1 - \text{fp}(\phi F_{n+1}),\end{aligned}$$

i.e., $1 - \text{fp}(k\phi) > \text{fp}(\phi F_{n+1})$, so we are done.

(2) $\phi F_{n+1} < F_{n+2}$. In this case, $\text{dist}(\phi F_{n+1}) = 1 - \text{fp}(\phi F_{n+1})$; thus, we need to show that $\text{fp}(\phi F_{n+1}) > \text{fp}(k\phi)$ and that $\text{fp}(\phi F_{n+1}) > 1 - \text{fp}(k\phi)$. The arguments are almost the same as in case (1), so we will not repeat them here. Again we find that $\text{dist}(\phi F_{n+1}) < \text{dist}(k\phi)$.

This completes the proof.

3. The New Characterization

Lemma 3.1: $[m\phi^2] + [n\phi^2]$ is never a Fibonacci number (m, n positive integers).

Proof: Suppose $[m\phi^2] + [n\phi^2] = F_i$ for some i. Since m and n are positive, $F_i \geq 5$ (just let $m = n = 1$). Now,

$$\left[\frac{F_i}{\phi^2}\right] = \left[\frac{F_i}{\phi}(\phi - 1)\right] = \left[F_i - \frac{F_i}{\phi}\right] = F_i - [F_i/\phi] - 1 = F_i - 1 - [\phi F_i - F_i]$$
$$= F_i - 1 - ([\phi F_i] - F_i),$$

which equals either $F_i - 1 - F_{i-1} = F_{i-2} - 1$ or $F_i - 1 - (F_{i-1} - 1) = F_{i-2}$. But we also have

$$\left[\frac{F_i}{\phi^2}\right] = \left[\frac{[m(1+\phi)] + [n(1+\phi)]}{(m+n)(1+\phi)} \cdot \frac{(m+n)(1+\phi)}{\phi^2}\right] =$$

$$= \left[\frac{m + n + [m\phi] + [n\phi]}{m + n + m\phi + n\phi} \cdot (m + n)\right].$$

To evaluate this expression, note that the denominator of the big fraction here exceeds the numerator by no more than 2, and the denominator is more than $2(m + n)$, so the fraction is greater than

$$\frac{2(m + n) - 2}{2(m + n)} = \frac{m + n - 1}{m + n}.$$

Hence, multiplying the fraction by $m + n$ will give a number between $m + n - 1$ and $m + n$, so the entire expression (after flooring) evaluates to $m + n - 1$.

Equating the two expressions for $[F_i/\phi^2]$, we see that $m + n - 1$ equals either $F_{i-2} - 1$ or F_{i-2}. In other words, there are two cases to be considered:

$$m + n = F_{i-2} \quad \text{and} \quad m + n = F_{i-2} + 1.$$

Suppose first that $m + n = F_{i-2}$. There are two subcases:

(1) $(m + n)\phi < F_{i-1}$. Then $[m\phi] + [n\phi]$ must equal either $F_{i-1} - 1$ or $F_{i-1} - 2$. But if $[m\phi] + [n\phi]$ were equal to $F_{i-1} - 2$, then $\text{fp}(m\phi) + \text{fp}(n\phi)$ would have to equal $1 + \text{fp}(\phi F_{i-2})$, so that either $\text{fp}(m\phi)$ or $\text{fp}(n\phi)$ would have to be greater than $\text{fp}(\phi F_{i-2})$. But $\text{fp}(\phi F_{i-2}) = 1 - \text{dist}(\phi F_{i-2})$, so this would mean that either m or n would not have the distance property, contradicting Lemma 2.1. Hence, $[m\phi] + [n\phi] = F_{i-1} - 1$.

(2) $(m + n)\phi > F_{i-1}$. Then $[m\phi] + [n\phi]$ must equal either $F_{i-1} - 1$ or F_{i-1}. But if $[m\phi] + [n\phi]$ were equal to F_{i-1}, we would have $\text{fp}(m\phi) + \text{fp}(n\phi) = \text{fp}(\phi F_{i-2})$, which would imply that either $\text{fp}(m\phi)$ or $\text{fp}(n\phi)$ was less than $\text{fp}(\phi F_{i-2})$. But $\text{fp}(\phi F_{i-2}) = \text{dist}(\phi F_{i-2})$, so this would mean that either m or n would not have the distance property, contradicting Lemma 2.1. So again we have $[m\phi] + [n\phi] = F_{i-1} - 1$.

It follows that

$$[m\phi^2] + [n\phi^2] = m + n + [m\phi] + [n\phi] = F_{i-2} + F_{i-1} - 1 = F_i - 1,$$

contrary to the assumption that $[m\phi^2] + [n\phi^2] = F_i$.

Suppose now that $m + n = F_{i-2} + 1$. Then $[m\phi] + [n\phi]$ is either $F_{i-1} + 1$ or F_{i-1}, and

$$[m\phi^2] + [n\phi^2] = m + n + [m\phi] + [n\phi] = F_{i-2} + 1 + F_{i-1} + r = F_i + 1 + r,$$

where $r = 0$ or 1, again contrary to the assumption that $[m\phi^2] + [n\phi^2] = F_i$. This establishes Lemma 3.1.

Lemma 3.2: If $[m\phi] + [n\phi]$ is a Fibonacci number (where m and n are distinct positive integers), then either $\text{fp}(m\phi) > \phi/2$ or $\text{fp}(n\phi) > \phi/2$, but not both.

Proof: Suppose $[m\phi] + [n\phi] = F_k$ for some k. Now $[(m + n)\phi]$ exceeds $[m\phi] + [n\phi]$ by at most one, so $[(m + n)\phi]$ is either F_k or $F_k + 1$. Let us write $[(m + n)\phi]$ as $F_k + r$, where $r = 0$ or 1. Then

$$(m + n)\phi - \text{fp}((m + n)\phi) = [(m + n)\phi] = F_k + r,$$

so

$$m + n = \frac{\text{fp}((m + n)\phi)}{\phi} + \frac{F_k + r}{\phi}$$

$$= \frac{\text{fp}((m + n)\phi)}{\phi} + \frac{r}{\phi} + \phi F_k - F_{k+1} + F_{k+1} - F_k$$

$$= \frac{\text{fp}((m + n)\phi)}{\phi} + \frac{r}{\phi} \pm \text{dist}(\phi F_k) + F_{k-1}.$$

Let x denote the sum of the first three terms in this expression. Since $\operatorname{dist}(\phi F_k) \le \frac{1}{2}$, we have $x > -1$. Moreover,

$$\frac{\operatorname{fp}((m+n)\phi)}{\phi} + \frac{r}{\phi} < \frac{2}{\phi},$$

so $x < (2/\phi) + \frac{1}{2} < 2$. It follows that $m + n$ equals either F_{k-1} or $F_{k-1} + 1$.

Suppose $m + n$ is F_{k-1}. Then $[(m+n)\phi] = [\phi F_{k-1}]$, which cannot equal $F_k + 1$ and, therefore, must equal F_k. Thus, $\operatorname{dist}(\phi F_{k-1}) = \operatorname{fp}(\phi F_{k-1})$. Then, by Lemma 2.1, $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ must both be greater than $\operatorname{fp}(\phi F_{k-1})$. But

$$\begin{aligned}[(m+n)\phi] - [m\phi] - [n\phi] &= \operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) - \operatorname{fp}((m+n)\phi)\\ &= \operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) - \operatorname{fp}(\phi F_{k-1})\end{aligned}$$

must be an integer, so it must equal 1. In other words,

$$[m\phi] + [n\phi] = [(m+n)\phi] - 1 = F_k - 1,$$

but this is a contradiction.

So $m + n = F_{k-1} + 1$. Then $m\phi + n\phi = \phi F_{k-1} + \phi$. We split into two cases:

(1) $\phi F_{k-1} > F_k$. We have

$$\begin{aligned}F_k = [m\phi] + [n\phi] &= m\phi + n\phi - \operatorname{fp}(m\phi) - \operatorname{fp}(n\phi)\\ &= \phi F_{k-1} + \phi - \operatorname{fp}(m\phi) - \operatorname{fp}(n\phi)\\ \Rightarrow \operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) &= \phi F_{k-1} - F_k + \phi\\ &= \operatorname{dist}(\phi F_{k-1}) + \phi.\end{aligned}$$

So it is clearly impossible for both $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ to be less than $\phi/2$; we need to show that they cannot both be greater than $\phi/2$. Suppose $\operatorname{fp}(m\phi) = \phi/2 + \varepsilon_1$ and $\operatorname{fp}(n\phi) = \phi/2 + \varepsilon_2$, where ε_1 and ε_2 are both positive and $\varepsilon_1 + \varepsilon_2 = \operatorname{dist}(\phi F_{k-1})$. Then $\operatorname{fp}(|m-n|\phi) = |\varepsilon_1 - \varepsilon_2| < \operatorname{dist}(\phi F_{k-1})$, but since m and n are distinct (and this is where distinctness is really crucial), $|m - n|$ is strictly positive, and this contradicts Lemma 2.1 (since $|m - n|$ is a positive integer less than F_{k-1}). Hence, $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ cannot both be greater than $\phi/2$.

(2) $\phi F_{k-1} < F_k$. The argument is similar, except that here

$$\operatorname{fp}(m\phi) + \operatorname{fp}(n\phi) = \phi - \operatorname{dist}(\phi F_{k-1}).$$

Then clearly we cannot have both $\operatorname{fp}(m\phi)$ and $\operatorname{fp}(n\phi)$ greater than $\phi/2$, and writing $\operatorname{fp}(m\phi) = \phi/2 - \varepsilon_1$ and $\operatorname{fp}(n\phi) = \phi/2 - \varepsilon_2$ leads to the same contradiction as before.

Lemma 3.3: If $\operatorname{fp}(m\phi) > \phi/2$, then $[m\phi] + [n\phi^2]$ is not a Fibonacci number. (m and n are positive integers but not necessarily distinct.)

Proof: We show that $F_k - [m\phi]$ can be written in the form $[n\phi]$ for some n if $F_k > [m\phi]$. There are two cases:

(1) $F_k = [\phi F_{k-1}] + 1$. If $F_{k-1} = m$, then $F_k - [m\phi] = 1$, which is of the form $[1 \cdot \phi]$. Otherwise, $F_{k-1} > m$. Moreover, by Lemma 2.1,

$$1 - \operatorname{fp}(m\phi) \ge \operatorname{dist}(m\phi) > \operatorname{dist}(\phi F_{k-1}) = 1 - \operatorname{fp}(\phi F_{k-1}),$$

i.e., $\operatorname{fp}(m\phi) < \operatorname{fp}(\phi F_{k-1})$. Let $d = (F_{k-1} - m)\phi$. Then

$$\operatorname{fp}(d) = \operatorname{fp}(\phi F_{k-1}) - \operatorname{fp}(m\phi) < 1 - \phi/2.$$

It follows that $[d + \phi] = [d] + 1$, and also that $[d] = [\phi F_{k-1}] - [m\phi]$. Thus,

$$F_k - [m\phi] = [\phi F_{k-1}] + 1 - [m\phi] = [d] + 1 = [d + \phi] = [(F_{k-1} - n + 1)\phi],$$

so we can just set $n = F_{k-1} - m + 1$.

(2) $F_k = [\phi F_{k-1}]$. Now the smallest value of m for which $\text{fp}(m\phi) > \phi/2$ is $m = 3$, so the smallest value of F_k for which this case can occur is $F_k = 8$. Since $\text{fp}(m\phi) = \text{dist}(m\phi)$, we have $\text{fp}(\phi F_{k-1}) < \text{fp}(5\phi) < 0.091$. So $\text{fp}((F_{k-1} - m)\phi) < 1 < \phi/2 + 0.091$; thus,

$$[(F_{k-1} - m + 1)\phi] = [(F_{k-1} - m)\phi] + 1.$$

Since $\text{fp}(\phi F_{k-1}) < \text{fp}(m\phi)$,

$$F_k - [m\phi] = [\phi F_{k-1}] - [m\phi] = [(F_{k-1} - m)\phi] + 1 = [(F_{k-1} - m + 1)\phi]$$

as we just proved, so we can just set $n = F_{k-1} - m + 1$, as before.

Now, since $1/\phi + 1/\phi^2 = 1$, we can apply Beatty's theorem, which states that $\{[na]\}$ and $\{[nb]\}$ form a partition of the positive integers if and only if a and b are irrational and $1/a + 1/b = 1$ (see [5], [6]). It follows that any number that can be written in the form $[n\phi]$ cannot be written in the form $[s\phi^2]$ for any s, so that Lemma 3.3 follows immediately.

Theorem 3.4: The two sets A and B of the Fibonacci-free partition can be written in the form

$$A = \{[n\phi]\} - \{[m\phi] \mid \text{fp}(m\phi) > \phi/2\},$$

$$B = \{[n\phi]\} \cup \{[m\phi] \mid \text{fp}(m\phi) > \phi/2\}.$$

Proof: First of all, we note that, by Beatty's theorem, A and B do indeed form a partition of the positive integers. From Lemmas 3.1-3.3, we see that this partition has the property that no two distinct integers from the same set sum to a Fibonacci number. The theorem then follows from the uniqueness of the Fibonacci-free partition.

4. Long's Conjecture

Lemma 4.1: If n is a positive integer such that $\text{fp}(n\phi) > \phi/2$, then there exists a positive integer k such that $n = [k\phi]$ and $\text{fp}(k\phi) < (\phi - 1)/2$.

Proof: First, note that

$$\text{fp}(n/\phi) = \text{fp}(n\phi - n) = \text{fp}(n\phi) > \phi/2.$$

Let $a = 1 - \text{fp}(n/\phi)$. Note that $a < 1 - \phi/2$. Then

$$\phi\lceil n/\phi\rceil = \phi(n/\phi) + a\phi = n + a\phi < n + (1 - \phi/2)\phi = n + (\phi - 1)/2.$$

Now set $k = \lceil n/\phi\rceil$. It is clear that k has the desired properties.

Lemma 4.2: If k is a positive integer such that $\text{fp}(k\phi) < (\phi - 1)/2$, then $\text{fp}([k\phi]\phi) > \phi/2$.

Proof:
$$\begin{aligned}\text{fp}([k\phi]\phi) &= \text{fp}(k\phi^2 - \phi\text{fp}(k\phi)) \\ &= \text{fp}(k\phi + k - \phi\text{fp}(k\phi)) \\ &= \text{fp}(k\phi - \phi\text{fp}(k\phi)) \\ &= \text{fp}(k\phi - \text{fp}(k\phi) - (\phi - 1)\text{fp}(k\phi)) \\ &= 1 - \text{fp}((\phi - 1)\text{fp}(k\phi)) \\ &> 1 - (\phi - 1)(\phi - 1)/2 \\ &= \phi/2.\end{aligned}$$

Lemma 4.3: If k is a positive integer such that $\text{fp}(k\phi) < (\phi - 1)/2$, then $[[k\phi]\phi] = [s\phi^3]$, where $s = [[k\phi](2 - \phi)]$.

Proof: By Lemma 4.2, $\text{fp}([k\phi]\phi) > \phi/2 = (2\phi + 1)/(2\phi + 2)$. Thus,

$$2(\phi + 1)\text{fp}([k\phi]\phi) > 2\phi + 1 \Rightarrow (1 + \phi^3)\text{fp}([k\phi]\phi) > \phi^3$$

$$\Rightarrow \text{fp}([k\phi]\phi) > \phi^3(1 - \text{fp}([k\phi]\phi)) = \phi^3\text{fp}([k\phi](2 - \phi))$$

$$\Rightarrow [[k\phi]\phi] = [[k\phi]\phi - \phi^3\text{fp}([k\phi](2 - \phi))].$$

Now

$$[k\phi]\phi = [k\phi](2 - \phi)\phi^3 = s\phi^3 + \phi^3\mathrm{fp}([k\phi](2 - \phi)).$$

Subtracting $\phi^3\mathrm{fp}([k\phi](2 - \phi))$ from both sides and then flooring both sides gives the required result.

Lemma 4.4: If n is a positive integer such that $\mathrm{fp}(n\phi) > \phi/2$, there exists a positive integer m such that $[n\phi] = [[m\phi]\phi^3]$ and $\mathrm{fp}(m\phi) < \phi/2$.

Proof: In view of Lemmas 4.1-4.3, we need to show that $[n(2 - \phi)]$ can be written in the form $[m\phi]$ with $\mathrm{fp}(m\phi) < \phi/2$. Now $[n(2 - \phi)] = [n(2\phi - 3)\phi]$. Let $m = \lceil n(2\phi - 3)\rceil$. We claim that this is the desired m. For

$$\begin{aligned} m\phi &= n(2\phi - 3)\phi + \phi(1 - \mathrm{fp}(n(2\phi - 3))) \\ &= n(2 - \phi) + \phi(1 - \mathrm{fp}(n(2\phi - 3))). \end{aligned}$$

Now $\mathrm{fp}(n(2\phi - 3)) = \mathrm{fp}(2n\phi) > \mathrm{fp}(\phi)$ so that $\phi(1 - \mathrm{fp}(n(2\phi - 3))) < \phi - 1$. Furthermore, $\mathrm{fp}(n(2 - \phi)) = 1 - \mathrm{fp}(n\phi) < 1 - \phi/2$. Thus,

$$\begin{aligned} n(2 - \phi) + \phi(1 - \mathrm{fp}(n(2\phi - 3))) &< [n(2 - \phi)] + 1 - \phi/2 + \phi - 1 \\ &= [n(2 - \phi)] + \phi/2. \end{aligned}$$

Hence, $[m\phi] = [n(2 - \phi)]$ and $\mathrm{fp}(m\phi) < \phi/2$, as required.

Lemma 4.5: If n is a positive integer such that $\mathrm{fp}(n\phi) < \phi/2$, there exists a positive integer m such that $[[n\phi]\phi^3] = [m\phi]$ and $\mathrm{fp}(m\phi) > \phi/2$.

Proof: First, we note that

$$\mathrm{fp}([n\phi]\phi) = 1 - \mathrm{fp}((\phi - 1)\mathrm{fp}(n\phi)) > 1 - (\phi - 1)\phi/2 = \frac{1}{2}.$$

(For a justification of the first equality in the above derivation, see the first five lines of the proof of Lemma 4.2 above.) Now

$$\mathrm{fp}([n\phi]\phi^3) = \mathrm{fp}([n\phi](2\phi + 1)) = \mathrm{fp}(2[n\phi]\phi).$$

Since $\mathrm{fp}([n\phi]\phi) > \frac{1}{2}$, it follows that $\mathrm{fp}(2[n\phi]\phi) = 2\mathrm{fp}([n\phi]\phi) - 1$. So we have $\mathrm{fp}([n\phi]\phi^3) = 2\mathrm{fp}([n\phi]\phi) - 1$. Thus,

$$(*)\qquad \begin{aligned} \mathrm{fp}([n\phi]\phi^3) + \phi(1 - \mathrm{fp}([n\phi]\phi)) &= 2\mathrm{fp}([n\phi]\phi) - 1 + \phi - \phi\mathrm{fp}([n\phi]\phi) \\ &= \phi - 1 + (2 - \phi)\mathrm{fp}([n\phi]\phi) \\ &< \phi - 1 + 2 - \phi = 1. \end{aligned}$$

Now let $m = [[n\phi]\phi^2]$. We claim that this is the desired m. For

$$\begin{aligned} [m\phi] &= [\phi([n\phi]\phi^2 + 1 - \mathrm{fp}([n\phi]\phi^2))] \\ &= [[n\phi]\phi^3 + \phi - \phi\mathrm{fp}([n\phi]\phi)] = [[n\phi]\phi^3]. \end{aligned}$$

The last equality follows from equation (*) above.

It remains to show that $\mathrm{fp}(m\phi) > \phi/2$. We have

$$[m\phi] = [[n\phi]\phi^3] = [[n\phi]\phi^2\phi] = [m\phi - \phi + \phi\mathrm{fp}([n\phi]\phi^2)].$$

But

$$\begin{aligned} -\phi + \phi\mathrm{fp}([n\phi]\phi^2) &= -\phi + \phi\mathrm{fp}([n\phi]\phi) \\ &= \phi(\mathrm{fp}([n\phi]\phi) - 1) > \phi\left(\frac{1}{2} - 1\right) = -\phi/2. \end{aligned}$$

It follows immediately that $\mathrm{fp}(m\phi) > \phi/2$, as required.

Theorem 4.6: The set A of the Fibonacci-free partition (defined in Theorem 3.4 above) satisfies the equality $A = \{[n\phi]\} - A'$, where $A' = \{[s\phi^3] \mid s \in A\}$.

Proof: From Lemma 4.4, we see that $\{[n\phi]\} - A' \subset A$, and from Lemma 4.5, we see that $A \subset \{[n\phi]\} - A'$.

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