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This is a proposed problem for the American Mathematical Monthly.

1 Problem Statement

Andrew and Barbara are competing in a curious Easter-egg hunt. There are n boxes numbered 1 to n, of which the organizers have randomly chosen k to put Easter eggs in. To play, a contestant submits a permutation $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. The smallest i for which the box numbered $\pi(i)$ contains an Easter egg is the contestant's score, and the contestant with the smallest score wins (ties are possible).

(a) Suppose Barbara learns what Andrew's permutation is before submitting her own. What permutation should she choose to maximize her chances of beating Andrew?

(b) Say that a permutation π is *fair* if, regardless of the value of k, π is equally likely to win or lose against the identity permutation. Show that π is fair if and only if it is *balanced*, meaning that for every i, $\pi(i) > i$ if and only if $\pi^{-1}(i) > i$.

(c) Deduce that if f_n is the number of fair permutations of n, and we adopt the convention that $f_0 = 1$, then

$$\sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = e^x \sec x.$$

(d) Suppose there are a thousand boxes arranged in ten rows of a hundred boxes each. Andrew decides to search the boxes row by row, starting with the top row, and searching from left to right within each row. Barbara decides to search the boxes column by column, starting with the leftmost column, and searching from top to bottom within each column. Which player is more likely to win?

2 Solution

(a) Assume without loss of generality that Andrew's permutation is the identity permutation. Then as long as $k \neq n$, Barbara's best choice is unique:

$$\pi := (\pi(1), \pi(2), \ldots, \pi(n)) = (2, 3, 4, \ldots, n, 1).$$

(Of course if $k = n$ then whatever permutation Barbara chooses, the outcome will be a tie.) With this choice of π, Barbara wins if and only if box 1 does *not* contain an Easter egg. Barbara cannot do better than this, because whenever box 1 *does* contain an Easter egg, then no matter what permutation Barbara chooses, she cannot do better than tie. Furthermore, in order to win whenever box 1 does *not* contain an Easter egg, Barbara must look in box 2 first in order to clinch the scenarios in which box 1 does not contain an Easter egg but box 2 does. Similarly, she must look in box 3 next, and so on down the line.

Note that the above argument is robust in the sense that it goes through even if the Easter eggs are not distributed uniformly.

(b) Let π be a permutation of $\{1,2,\ldots,n\}$. Our proof uses the concept of the *π-transform* $\hat{S}$ of a subset $S \subseteq \{1,2,\ldots,n\}$, which we now define. First, let

$$\Sigma := \{(i,0) : \pi(i) = i\} \cup \{(i,-1) : i < \pi(i)\} \cup \{(i,+1) : \pi^{-1}(i) > i\}.$$

(Since ordered pairs are notationally clumsy, from now on we write i_0, i_+, and i_- instead of $(i,0)$, $(i,+1)$ and $(i,-1)$.) Equivalently, Σ contains the symbol i_0 if i is a *fixed point* of π (i.e., $\pi(i) = i$), the symbol i_- if i is a *upgrade* of π (i.e., $\pi^{-1}(i) < i < \pi(i)$), the symbol i_+ if i is a *downgrade* of π (i.e., $\pi^{-1}(i) > i > \pi(i)$), and both the symbols i_- and i_+ if i is a *valley* of π (i.e., $\pi^{-1}(i) > i < \pi(i)$).

Given $S \subseteq \{1,2,\ldots,n\}$, we define $\hat{S}$ to be the following subset of Σ:

$$\begin{aligned}\hat{S} := \{i_0 : i \in S \text{ and } \pi(i) = i\} &\cup \{i_- : i \in S \text{ and } i < \pi(i)\} \\ &\cup \{i_+ : \pi^{-1}(i) \in S \text{ and } \pi^{-1}(i) > i\}.\end{aligned}$$

In words, we take each element $i \in S$ in turn and consider $j := \pi(i)$; if $i = j$, then we put i_0 in $\hat{S}$; if $i < j$, then we put i_- in $\hat{S}$; if $i > j$, then we put j_+ (note: *not* i_+) in $\hat{S}$. It is easy to check that the map $S \mapsto \hat{S}$ is a cardinality-preserving bijection from the set of all subsets of $\{1,2,\ldots,n\}$ to the set of all subsets of Σ.

The point of the π-transform is that if S is the set of boxes containing Easter eggs, then we can easily read off from $\hat{S}$ whether π wins or loses or ties against the identity permutation. Intuitively, $\hat{S}$ "remembers," for each $i \in S$, whether i or $\pi(i)$ is smaller, and this information is all we need. More precisely, let i be the smallest number *appearing* in $\hat{S}$ (meaning that $i_0 \in \hat{S}$ or $i_- \in \hat{S}$ or $i_+ \in \hat{S}$). Then one readily verifies that

1. if $i_0 \in \hat{S}$, or if *both* i_- and i_+ are in $\hat{S}$, then π ties;
2. if $i_- \in \hat{S}$ but $i_+ \notin \hat{S}$, then π wins;
3. if $i_+ \in \hat{S}$ but $i_- \notin \hat{S}$, then π loses.

Let us now assume that π is balanced and argue that π is fair. It follows from the definition that a balanced permutation contains no upgrades or downgrades. Now fix any k. We give a bijection ϕ between the set $\mathcal{W}_k$ of π-transforms of winning k-element subsets and the set $\mathcal{L}_k$ of π-transforms of losing k-element subsets. If $\hat{S} \in \mathcal{W}_k$, then the smallest number i appearing in $\hat{S}$ appears as i_- but not as i_+. So let $\phi(\hat{S})$ be the set we obtain from $\hat{S}$ by replacing i_- with i_+. This definition makes sense because i must be a valley, and hence Σ contains both i_- and i_+. It is easy to see that ϕ is a bijection (a "sign-reversing involution" if you like) between $\mathcal{W}_k$ and $\mathcal{L}_k$.

Conversely, suppose that π is not balanced. Then π must contain a upgrade or a downgrade. Let i be the smallest upgrade or downgrade. If there are f

fixed points and v valleys smaller than i, then let $k = n - f - 2v$. We claim that $|\mathcal{W}_k| \neq |\mathcal{L}_k|$. If $\hat{S}$ is a k-element subset of Σ that does not yield a tie, then the smallest number i' appearing in $\hat{S}$ is either i or a valley that is smaller than i. For any valley z, the wins and losses with $i' = z$ are equinumerous, by the same sign-reversing-involution argument ($i'_- \leftrightarrow i'_+$) as before. All that remains is the unique k-element set $\hat{S}$ for which $i' = i$, and this $\hat{S}$ yields either a win or a loss (according to whether i is a upgrade or a downgrade), and not a tie. So for this value of k, the total wins and losses are not equinumerous; that is, π is not fair.

(c) It is easy to see that if π is a balanced permutation, then each cycle of π is either a fixed point or an *alternating* cycle of even length, meaning that if z is the largest element of the cycle, then

$$z > \pi(z) < \pi(\pi(z)) > \pi(\pi(\pi(z))) < \cdots < z.$$

By deleting z, we see that the number of ways to arrange a fixed set of $2m$ elements into an alternating cycle of length $2m$ equals the number of alternating *permutations* σ of length $2m - 1$, by which we mean that

$$\sigma(1) < \sigma(2) > \sigma(3) < \cdots > \sigma(2m - 1).$$

If a_{2m-1} is the number of alternating permutations of length $2m - 1$, then it is a standard result [1, page 149] that

$$\sum_{m=1}^{\infty} a_{2m-1} \frac{x^{2m-1}}{(2m-1)!} = \tan x.$$

Therefore the exponential generating function for alternating cycles of even length is

$$\sum_{m=1}^{\infty} a_{2m-1} \frac{x^{2m}}{(2m)!} = \int \tan x \, dx = \ln \sec x.$$

So the exponential generating function for the kinds of cycles appearing in a balanced permutation is $x + \ln \sec x$ (the x accounts for fixed points), and by standard generatingfunctionology, the exponential generating function for balanced permutations is therefore $\exp(x + \ln \sec x) = e^x \sec x$.

We remark that if $n = 2m$ is even, then the number of fair permutations is twice the number of *Salié permutations*, which are permutations σ such that for some $r \leq m$,

$$\sigma(1) < \sigma(2) > \sigma(3) < \cdots < \sigma(2r)$$

and

$$\sigma(2r) < \sigma(2r + 1) < \sigma(2r + 2) < \cdots < \sigma(2m).$$

It would interesting to construct an explicit 2-to-1 map from fair permutations to Salié permutations.

(d) Andrew is more likely to win, unless $k = 1$ or $981 < k \leq 1000$, in which case both players are equally likely to win.

We number the boxes so that Andrew's search order is given by the identity permutation, but instead of numbering the boxes from 1 to 1000, we number them from 000 to 999. That is, the boxes are labeled with three-digit base-ten numbers abc. With this notation, Barbara's permutation π is simply a right circular shift; i.e., $\pi(abc) = cab$. Then we see that abc is an upgrade if and only if $c \geq a > b$, and abc is a downgrade if and only if $c \leq a < b$.

If i is a downgrade (respectively, an upgrade), then let $\mathcal{A}_i$ (respectively, $\mathcal{B}_i$) be the set of k-element subsets $\hat{S} \subseteq \Sigma$ such that i is the smallest number appearing in $\hat{S}$. The sets $\mathcal{A}_i$ (respectively, $\mathcal{B}_i$) are winners for Andrew (respectively, Barbara). As explained in the solution to part (b) above, all other k-element subsets of Σ either tie or cancel out by a sign-reversing involution, so to show that Andrew's chances are at least as good as Barbara's, it suffices to exhibit a bijective map ϕ from the set of downgrades to the set of upgrades such that $|\mathcal{A}_i| \geq |\mathcal{B}_{\phi(i)}|$ for all downgrades i.

If abc is a downgrade, then let $\phi(abc) = a'b'c'$ where $a' = a+1$, $b' = c$, and $c' = b$. Note that $a' \leq 9$ because $a < b \leq 9$, so $a'b'c'$ is indeed a three-digit number. Moreover, $c' \geq a' > b'$ so $\phi(abc)$ is an upgrade. Similarly, given an upgrade, we can decrement the leading digit and swap the two remaining digits to obtain a downgrade; this map is clearly the inverse of ϕ, so ϕ is indeed a bijection as claimed.

By construction, ϕ has the property that $i < \phi(i)$ for all downgrades i. It follows that $|\mathcal{A}_i| \geq |\mathcal{B}_{\phi(i)}|$ for all downgrades i, because the size of $\mathcal{A}_i$ (or $\mathcal{B}_i$) is just $\binom{m}{k-1}$ where m is the number of elements of Σ larger than i. Finally, note that 000 is a fixed point and that $001, 002, \ldots, 009$ are valleys, so there are 20 elements of Σ less than or equal to the upgrade 010. Hence

$$|\mathcal{A}_{010}| = \binom{1000-20}{k-1} = \binom{980}{k-1},$$

which is *strictly* greater than $|\mathcal{B}_{\phi(010)}| = |\mathcal{B}_{101}|$ as long as $0 < k-1 \leq 980$. Thus Andrew has an advantage over Barbara provided $1 < k \leq 981$. On the other hand if $k > 981$ then there are no sets $\mathcal{A}_i$ or $\mathcal{B}_i$, and if $k = 1$ then $|\mathcal{A}_i| = |\mathcal{B}_{\phi(i)}| = 1$ for all downgrades i; in either case, Andrew and Barbara have equal chances of winning.

References

[1] Richard P. Stanley, *Enumerative Combinatorics, vol. 1*, Cambridge University Press, 1997.