

The Erasing Marks Conjecture (Timothy Y. Chow, December 2019)

The Stanley–Stembridge e -positivity conjecture for $(\mathbf{3} + \mathbf{1})$ -free posets is still open as of this writing but has been proved in some special cases. In particular, in his 1995 paper, Stanley already proved it for a graph that is a path. However, even in this case, I know of no explicit description of a basis for equivariant cohomology with the property that the dot action simply permutes the basis elements. An inductive construction can be extracted from C. Procesi’s paper, “The toric variety associated to Weyl chambers” (*Mots*, Lang. Raison. Calc., Hermès, Paris, 1990, pp. 153–161), but it is not very explicit.

In 2015, I formulated (a slightly messier version of) the conjecture below. **NOTE ADDED IN LATE 2020: The conjecture has been proved by Cho–Hong–Lee, arXiv:2008.12500.**

Let S_n denote the symmetric group.

Definition. A *mark set* is a subset $m := \{m_1, m_2, \dots, m_k\} \subset \{1, 2, \dots, n-1\}$. By convention we assume that $m_1 < m_2 < \dots < m_k$.

Definition. Given a mark set m , let $Y(m)$ denote the Young subgroup

$$S_{m_1} \times S_{m_2 - m_1} \times S_{m_3 - m_2} \times \dots \times S_{n - m_k}.$$

Let $Y(m)$ act on S_n on the right (i.e., on positions). Call the orbits of this action *m -orbits*.

Example. Let $n = 9$ and let $m = \{1, 2, 4, 7, 8\}$. If we write elements of S_n in one-line notation then we can visualize each m_i as “marking” the space between the m_i th and the $(m_i + 1)$ st letter (with a vertical bar, say), and we can visualize $Y(m)$ as permuting elements between marks. So the following elements of S_n are in the same m -orbit:

$$3|6|85|947|1|2 \sim 3|6|58|497|1|2$$

Definition. If $\pi \in S_n$ and m is a mark set, let $p(\pi, m)$ be the polynomial in t defined by the formula

$$p(\pi, m) := \prod_{i=1}^k (t_{\pi(m_i)} - t_{\pi(m_i+1)}).$$

Definition. If m is a mark set then its *erasure* $e(m)$ is the mark set defined by

$$e(m) := \{i \in m : i \neq 1 \text{ and } i - 1 \notin m\}.$$

Erasing Marks Conjecture. For each mark set m and each $e(m)$ -orbit C , define an equivariant cohomology class of a regular semisimple permutahedral Hessenberg variety (i.e., $h_i = i + 1$ for all i) by putting (on the moment graph) $p(\pi, m)$ at every $\pi \in C$, and putting 0 at every other vertex. Then these classes form a basis for equivariant cohomology.

Remark. The conjecture explicitly associates a tabloid representation to each acyclic orientation O of a path with n vertices. Number the edges $1, 2, \dots, n - 1$ from left to right, and let m be the set of edges that are oriented from left to right. The cardinality of m is the number of ascents, and it is not hard to show that the cardinality of $e(m)$ is the number of sinks minus one. If λ is the type of the Young subgroup $Y(e(m))$ then we associate to O the tabloid representation M^λ . Note that $\ell(\lambda)$ equals the number of sinks.

We now show, using a result of J. R. Stembridge, “Eulerian numbers, tableaux, and the Betti numbers of a toric variety” (*Discrete Math.* **99** (1992), 307–320), that the conjectured basis has the correct cardinality $n!$, so it “only” remains to show that the vectors are a spanning set.

Definition. For any finite sequence α of nonnegative integers, let its *support* $S(\alpha)$ be the set of positive integers occurring in α . Say that α is *admissible* if either $S(\alpha)$ is empty or $S(\alpha) = \{1, 2, \dots, k\}$ for some positive integer k .

Definition. A *code* is an ordered pair (α, f) , where α is an admissible finite sequence of nonnegative integers, and f is a integer-valued function on $S(\alpha)$ such that for all $i \in S(\alpha)$, $1 \leq f(i) < \mu_i$, where μ_i is the number of occurrences of i in α .

If (α, f) is a code, note that $\mu_i \geq 2$ for all $i \in S(\alpha)$; i.e., every positive integer must occur at least twice in α if it appears at all. It is convenient to represent a code by writing down α and then, for each $i \in S(\alpha)$, putting a hat on the $(f(i) + 1)$ st occurrence of i .

Example. The codes of length 3 are 000, 01 $\hat{1}$, 10 $\hat{1}$, $\hat{1}10$, $\hat{1}\hat{1}1$, 11 $\hat{1}$.

Example. The codes of length 4 are 0000, 001 $\hat{1}$, 010 $\hat{1}$, 01 $\hat{1}0$, 100 $\hat{1}$, 10 $\hat{1}0$, $\hat{1}\hat{1}00$, 01 $\hat{1}\hat{1}$, 10 $\hat{1}\hat{1}$, $\hat{1}\hat{1}01$, $\hat{1}\hat{1}10$, 011 $\hat{1}$, 101 $\hat{1}$, 110 $\hat{1}$, 11 $\hat{1}0$, $\hat{1}\hat{1}\hat{1}1$, $\hat{1}\hat{1}\hat{1}1$, $\hat{1}\hat{1}\hat{1}\hat{1}$, $\hat{1}\hat{1}\hat{2}\hat{2}$, $\hat{1}\hat{2}\hat{1}\hat{2}$, $\hat{1}\hat{2}\hat{2}\hat{1}$, $\hat{2}\hat{1}\hat{1}\hat{2}$, $\hat{2}\hat{1}\hat{2}\hat{1}$.

The following theorem is an immediate corollary of Theorem 1.1 in the aforementioned paper by Stembridge.

Theorem. The number of codes of length n is $n!$.

If (α, f) is a code, say that (α', f) is a *shuffle* of (α, f) if α' is a reordering of α . Note that every shuffle of a code is also a code. We say that two codes are in the same *shuffle equivalence class* if one is a shuffle of the other. If, as above, we let μ_i denote the number of occurrences of i in (α, f) , then the number of codes in the shuffle equivalence class of (α, f) is the multinomial coefficient $\binom{n}{\mu_0, \mu_1, \mu_2, \dots}$.

In the setting of the Erasing Marks Conjecture, let m be a mark set and let $e(m) = \{e_1, \dots, e_r\}$ be its erasure, where $e_1 < e_2 < \dots < e_r$. (If $e(m)$ is empty then $r = 0$.) By convention, set $e_0 = 0$ and $e_{r+1} = n$. Define the *block lengths* $\ell_0, \ell_1, \dots, \ell_r$ of $e(m)$ by $\ell_i := e_{r-i+1} - e_{r-i}$. (This “backward” numbering of the block lengths may look unnatural but it will simplify things later.) Then the number of $e(m)$ -orbits is the multinomial coefficient $\binom{n}{\ell_0, \ell_1, \ell_2, \dots}$.

Therefore, to show that our conjectured basis has cardinality $n!$, it suffices to exhibit a bijection from mark sets m to shuffle equivalence classes of codes with the property that the block sizes of $e(m)$ coincide with the multiplicities of the integers in the code.

Before describing the bijection, we make some preliminary observations. A shuffle equivalence class of codes is uniquely determined by the values of μ_i and the function f . Conversely, if we specify any nonnegative integers $\mu_0, \mu_1, \mu_2, \dots$ that sum to n , along with positive integers $f(i)$ satisfying $1 \leq f(i) < \mu_i$ for all $i \geq 1$, then there exists some shuffle equivalence class that gives rise to these values. Thus we may identify a shuffle equivalence class with a set of values of μ_i and $f(i)$ with these properties.

Next, let us compare the structures of m and $e(m)$. Note first that $\ell_i \geq 2$ for $1 \leq i \leq r$, because if m contains consecutive integers then the larger one gets “erased” when we pass to $e(m)$. For similar reasons, if $1 \leq i \leq r$, then between e_{r-i} and e_{r-i+1} there can be at most $\ell_i - 2$ elements of m , and any such elements of m must comprise a consecutive sequence $e_{r-i} + 1, e_{r-i} + 2, e_{r-i} + 3, \dots$. The case $i = 0$ is special; any elements of m larger than e_r must still comprise a consecutive sequence $e_r + 1, e_r + 2, e_r + 3, \dots$, but there can be up to $\ell_0 - 1$ such elements of m .

It follows that a mark set m is uniquely characterized by $e(m)$ together with integers j_0, j_1, \dots, j_r , where $j_0 \leq \ell_0 - 1$ and $j_i \leq \ell_i - 2$ for $1 \leq i \leq r$. The number j_i represents the number of elements of m between e_{r-i} and e_{r-i+1} . Moreover, if we choose any positive integers $\ell_0, \ell_1, \dots, \ell_r$ summing to n such that $\ell_i \geq 2$ for $1 \leq i \leq r$, and we specify any nonnegative integers j_0, j_1, \dots, j_r such that $j_0 \leq \ell_0 - 1$ and $j_i \leq \ell_i - 2$ for $1 \leq i \leq r$, then there is some mark set that gives rise to these numbers.

We see that the data needed to specify a shuffle equivalence class is very similar to the data needed to specify a mark set (think $\mu \leftrightarrow \ell$ and $f \leftrightarrow j + 1$), except that μ_0 can be zero while $\ell_0 \geq 1$, and there is no such thing as $f(0)$ whereas there does exist j_0 .

To describe the desired bijection from mark sets to shuffle equivalence classes, we split into two cases.

Case 1. $j_0 = 0$. To specify the corresponding shuffle equivalence class, we set $\mu_i := \ell_i$ for $0 \leq i \leq r$. For $1 \leq i \leq r$, set $f(i) := j_i + 1$.

Case 2. $j_0 \geq 1$. To specify the corresponding shuffle equivalence class, set $\mu_0 := 0$. For $1 \leq i \leq r$, set $\mu_i := \ell_{i-1}$. Set $f(1) := j_0$, and for $2 \leq i \leq r$, set $f(i) := j_{i-1} + 1$.

It is straightforward to verify that this does indeed yield a bijection. Moreover, we have the desired equality $\binom{n}{\ell_0, \ell_1, \ell_2, \dots} = \binom{n}{\mu_0, \mu_1, \mu_2, \dots}$.

Example. For $n = 3$, the correspondence is shown below.

$$\begin{array}{rcl}
 \ell_0 = 3, & j_0 = 0 & \leftrightarrow \mu_0 = 3 \\
 \ell_0 = 1, & \ell_1 = 2, & j_0 = 0, \quad j_1 = 0 \quad \leftrightarrow \mu_0 = 1, \quad \mu_1 = 2, \quad f(1) = 1 \\
 & & \ell_0 = 3, \quad j_0 = 1 \quad \leftrightarrow \mu_0 = 0, \quad \mu_1 = 3, \quad f(1) = 1 \\
 & & \ell_0 = 3, \quad j_0 = 2 \quad \leftrightarrow \mu_0 = 0, \quad \mu_1 = 3, \quad f(1) = 2
 \end{array}$$