

A Note on a Combinatorial Interpretation of the e -Coefficients of the Chromatic Symmetric Function

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Abstract. Stanley has studied a symmetric function generalization X_G of the chromatic polynomial of a graph G . The innocent-looking Stanley-Stembridge Poset Chain Conjecture states that the expansion of X_G in terms of elementary symmetric functions has nonnegative coefficients if G is a clawfree incomparability graph. Here we present a new approach that combines Gasharov's work on the conjecture with Egecioglu and Remmel's combinatorial interpretation of the inverse Kostka matrix. An interesting byproduct of our arguments is a connection, apparently new, between acyclic orientations and P -tableaux.

1. INTRODUCTION

The main ideas in this note are simple but require an inordinate number of definitions to state. In this section we skip most of these definitions so as not to obscure the exposition with a mass of technicalities. The missing definitions are given in the next section.

Let G be a finite simple undirected graph and let X_G be its chromatic symmetric function. Expand X_G in terms of elementary symmetric functions e_λ and call the coefficients a_λ^G :

$$X_G = \sum_{\lambda} a_\lambda^G e_\lambda.$$

One of the outstanding conjectures about X_G (and the motivation for this note) is the Stanley-Stembridge *Poset Chain Conjecture* [6]: if G is a clawfree incomparability graph, then $a_\lambda^G \geq 0$ for all λ .

It is natural to attack this conjecture by looking for a combinatorial interpretation of a_λ^G . We can obtain such an interpretation as follows. Observe first that Gasharov [2] tells us that if G is a clawfree incomparability graph, then the coefficients of the Schur-function expansion of X_G have a combinatorial interpretation. Next, observe that to convert from the Schur-function expansion of X_G to the e -expansion of X_G , we need to introduce an inverse Kostka matrix. But Egecioglu and Remmel [1] have a combinatorial interpretation of the inverse Kostka matrix. Combining these two combinatorial interpretations therefore gives us a combinatorial interpretation of the coefficients a_λ^G .

This simple observation does not in itself prove the Poset Chain Conjecture, because Egecioglu and Remmel's combinatorial interpretation (and therefore our combinatorial

interpretation of a_λ^G) involves a *signed* sum over combinatorial objects. However, it does open up a new line of attack on the Poset Chain Conjecture: we can try to prove the nonnegativity of a_λ^G by looking for sign-reversing involutions. This is illustrated below by a new proof of the fact (first shown by Stanley [5, Theorem 3.3]) that if G is a clawfree incomparability graph, then for all ℓ ,

$$\sum_{\lambda: \ell(\lambda)=\ell} a_\lambda^G$$

is the number of acyclic orientations of G with ℓ sinks. This new proof is not significantly shorter than Stanley's, but in addition to using completely different methods, in my opinion it requires significantly less ingenuity; the sign-reversing involution is very simple and one of the first things one might try. This provides hope that more sophisticated involutions will produce correspondingly stronger results.

An interesting byproduct of the proof is Lemma 1 below, which describes a connection, apparently not previously noticed, between acyclic orientations and P -tableaux.

2. BACKGROUND

We now provide the necessary technical background. Some familiarity with the basics of symmetric functions and partitions is assumed; see [4] or [3, Chapter I].

For the expert, we remark that there are two points where we diverge slightly from the literature: we use English style for our Ferrers diagrams while Egecioglu and Remmel use French style, and by " P -tableau" we mean the *transpose* of what Gasharov calls a P -tableau.

Let G be a finite simple undirected graph with vertex set $V = \{v_1, v_2, \dots, v_d\}$. A *proper coloring* of G is a map $\kappa : V \rightarrow \mathbb{N}$ such that $\kappa(v_i) \neq \kappa(v_j)$ whenever v_i and v_j are adjacent. Let $\{x_n \mid n \in \mathbb{N}\}$ be a countably infinite family of independent indeterminates. Following [5], define the *chromatic symmetric function* X_G of G to be the formal power series

$$X_G \stackrel{\text{def}}{=} \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where the sum is over all proper colorings of G . It is easy to see that X_G is a symmetric function, so it can be written as a (finite) linear combination of elementary symmetric functions e_λ :

$$X_G = \sum_{\lambda \vdash d} a_\lambda^G e_\lambda.$$

As we mentioned in the introduction, one of the main open problems in this area is the *Poset Chain Conjecture* of Stanley and Stembridge [6]. This states that if G is a

clawfree incomparability graph, then G is e -positive, i.e., $a_\lambda^G \geq 0$ for all λ . Recall that an *incomparability graph* is a graph obtained from a finite poset by letting the vertex set of the graph be the vertex set of the poset and connecting two vertices with an edge if and only if the vertices are incomparable elements in the poset. *Clawfree* just means that the graph does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. Clawfree incomparability graphs are also referred to as “incomparability graphs of $(\mathbf{3} + \mathbf{1})$ -free posets.” A poset is $(\mathbf{3} + \mathbf{1})$ -free if it does not contain an induced subposet isomorphic to a disjoint union of a three-element chain and a one-element chain. It is clear that an incomparability graph of a poset is clawfree if and only if the poset is $(\mathbf{3} + \mathbf{1})$ -free.

One of the most important partial results towards the Poset Chain Conjecture is due to Gasharov. If P is a finite poset, then define a P -tableau to be an arrangement of the elements of P into a Ferrers shape (English style) such that

1. each element of P is used exactly once,
2. if x appears immediately above y in a column then $x \prec y$, and
3. if x appears immediately to the left of y in a row then $x \not\prec y$.

(We have chosen to use the transpose of Gasharov’s definition of P -tableaux both for convenience in our proof and because in general column-strict tableaux are more commonly used in the literature than row-strict tableaux.) Next, define f_λ^G by the equation

$$\omega X_G = \sum_{\lambda \vdash d} f_\lambda^G s_\lambda,$$

i.e., expand ωX_G in terms of Schur functions and let f_λ^G be the coefficient of s_λ . (Here ω is the involution that sends s_λ to $s_{\lambda'}$.) We can now state Gasharov’s result [2].

PROPOSITION 1. *If P is a $(\mathbf{3} + \mathbf{1})$ -free poset and G is its incomparability graph, then f_λ^G is the number of P -tableaux of shape λ .*

The last piece of background is Egecioglu and Remmel’s combinatorial interpretation of the inverse Kostka matrix. A *special rim hook tabloid* T of shape μ and type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a filling of the Ferrers diagram of μ repeatedly with rim hooks (a.k.a. *skew hooks* or *border strips* or *ribbons*) of sizes $\{\lambda_1, \dots, \lambda_\ell\}$ such that each rim hook is *special*, by which we mean that it contains at least one cell in the first column. In other words, to create a special rim hook tabloid, take any rim hook that contains at least one cell in the first column and that leaves a legal Ferrers diagram when removed; then remove this rim hook and repeat the process iteratively on the residue. Note that only the *sizes* of the rim hooks matter and not their *order*, in contrast to the usual notion of a rim hook tableau. In other words, we may if we wish peel off a large rim hook first, then a small one, then a large one, and so on, so long as in the end we have the right number of rim hooks of each size. For this reason, special rim hook tabloids are typically drawn not by putting

numbers in the boxes of the Ferrers diagram but by connecting the boxes in question with a continuous zigzag line.

The *sign* $\text{sgn} T$ of a special rim hook tabloid T is defined in the expected way: the sign of a rim hook is $(-1)^{h-1}$ where h is the height of the rim hook, and the sign of T is the product of the signs of its component rim hooks.

Eğecioğlu and Remmel's result is the following [1].

PROPOSITION 2. *The inverse Kostka matrix $K_{\lambda,\mu}^{-1}$ satisfies*

$$K_{\lambda,\mu}^{-1} = \sum_T \text{sgn} T,$$

where the summation is over all special rim hook tabloids of type λ and shape μ .

3. A COMBINATORIAL INTERPRETATION OF a_λ^G

Throughout this section, unless otherwise stated, P will be a $(\mathbf{3} + \mathbf{1})$ -free poset, G will be its incomparability graph, and X_G will be the chromatic symmetric function of G . The coefficients a_λ^G and f_λ^G are defined as above.

It is well known (e.g., [3, §I.6, Table 1]) that the change-of-basis matrix between the Schur functions and the complete homogeneous symmetric functions is the inverse Kostka matrix, i.e.,

$$s_\mu = \sum_\lambda K_{\lambda,\mu}^{-1} h_\lambda.$$

Therefore

$$\sum_\lambda a_\lambda^G h_\lambda = \omega X_G = \sum_\mu f_\mu^G s_\mu = \sum_\mu f_\mu^G \sum_\lambda K_{\lambda,\mu}^{-1} h_\lambda = \sum_\lambda \left(\sum_\mu K_{\lambda,\mu}^{-1} f_\mu^G \right) h_\lambda,$$

i.e.,

$$a_\lambda^G = \sum_\mu K_{\lambda,\mu}^{-1} f_\mu^G.$$

Now let us invoke Proposition 2. We obtain

$$a_\lambda^G = \sum_\mu \sum_T (\text{sgn} T) f_\mu^G, \tag{3.1}$$

where the inner sum is over all special rim hook tabloids of type λ and shape μ .

The right-hand side of (3.1) suggests the following definition. A *special rim hook P -tableau* T of type λ and shape μ is an ordered pair (T', T'') where T' is a P -tableau of

shape μ and T'' is a special rim hook tabloid of type λ and shape μ . The *sign* of T is just the sign of T'' . Note that because T and T' have the same shape, we may visualize a special rim hook P -tableau as an ordinary P -tableau equipped with a decomposition into special rim hooks, i.e., we need not visualize two separate tableaux.

We can now state our combinatorial interpretation of a_λ^G . Combining Proposition 1 and (3.1) yields

THEOREM 1. *If G is the incomparability graph of a $(\mathbf{3} + \mathbf{1})$ -free poset P , then*

$$a_\lambda^G = \sum_T \operatorname{sgn} T,$$

where the sum is over all special rim hook P -tableaux T of type λ .

To illustrate the power of Theorem 1, we use it to prove

PROPOSITION 3. *If G is the incomparability graph of a $(\mathbf{3} + \mathbf{1})$ -free poset P , then for all ℓ ,*

$$\sum_{\lambda: \ell(\lambda)=\ell} a_\lambda^G$$

is the number of acyclic orientations of G with exactly ℓ sinks.

Recall that an *acyclic orientation* of G is an assignment of a direction to each edge of G in such a way that no directed cycles are formed. Stanley [5, Theorem 3.3] originally proved Proposition 3 with no restriction on G . For our proof, we need the following lemma.

LEMMA 1. *Let G be the incomparability graph of an arbitrary finite poset P . Let κ_ℓ be the number of acyclic orientations of G with exactly ℓ sinks and let π_k be the number of P -tableaux whose shape is a hook with k cells in the first column. Then*

$$\pi_k = \sum_\ell \binom{\ell-1}{k-1} \kappa_\ell.$$

Proof. Let T be a P -tableau whose shape is a hook with k cells in the first column. Then T induces an acyclic orientation of G as follows: if u and v are connected by an edge in G , then we make u point towards v if, in T , the column that u is in lies to the right of the column that v is in, and we make v point towards u if the column that u is in lies to the left of the column that v is in. Note that u and v cannot be in the same column, for then they would be comparable in P and therefore non-adjacent in G . It is clear that this orientation is acyclic.

Let \mathfrak{o} be an acyclic orientation of G with ℓ sinks. We claim that for all k , \mathfrak{o} is induced by exactly $\binom{\ell-1}{k-1}$ P -tableaux whose shape is a hook with k cells in the first column. This will prove the lemma.

Suppose we are given \mathfrak{o} and k . Think of \mathfrak{o} as a poset, with $u < v$ if and only if there is a directed path from v to u . To avoid confusing this partial ordering with the partial ordering of P , we use \prec to denote the order relation of the latter. We construct a P -tableau T as follows. Consider the ℓ sinks of \mathfrak{o} , i.e., the ℓ minimal elements of \mathfrak{o} . These are mutually non-adjacent in G and therefore they form a chain in P . Let the \prec -minimal element of this chain be the $(1, 1)$ cell of T (i.e., the cell in the first row and the first column of T). Choose $k - 1$ out of the remaining $\ell - 1$ sinks and arrange these in \prec -ascending order down the first column of T . Arrange the remaining elements along the first row of T as follows. At each stage, the remaining elements form an induced subposet of \mathfrak{o} . The minimal elements of this subposet form a chain in P . Choose the \prec -minimal element of this chain to be the next element in the first row of T , and repeat this process until all elements have been placed.

We must check that T is a P -tableau. We need only check that if u and v are consecutive elements in the first row, then $u \not\succeq v$. If $u \succ v$, then in particular u and v are \prec -comparable and therefore non-adjacent in G . There are two cases.

Case 1: u is not the element in the $(1, 1)$ cell of T . Then u is a \prec -minimal element of some induced subposet \mathfrak{o}' of \mathfrak{o} and v is a \prec -minimal element of $\mathfrak{o}' \setminus u$. Now \prec -minimal elements of $\mathfrak{o}' \setminus u$ are either \prec -minimal elements of \mathfrak{o}' or else elements that cover u (in the \prec -ordering). However, v cannot cover u because v and u are non-adjacent in G . Hence v is a \prec -minimal element of \mathfrak{o}' , and since u is \prec -minimal among all \prec -minimal elements of \mathfrak{o}' , we must have $u \prec v$, a contradiction.

Case 2: u is in the $(1, 1)$ cell of T . Then v is a \prec -minimal element of the subposet \mathfrak{o}' obtained by deleting all elements in the first column of T from \mathfrak{o} . Again v is either a \prec -minimal element of \mathfrak{o} or else v covers one of the elements in the first column of T . But as in Case 1, v cannot cover u , and if u' is some other element in the first column of T , then $v \prec u \prec u'$ so v cannot cover u' either. Hence, arguing as before, v is a \prec -minimal element of \mathfrak{o} and $u \prec v$, a contradiction.

Thus T is indeed a P -tableau. It is easy to see that T induces \mathfrak{o} . The $\binom{\ell-1}{k-1}$ P -tableaux produced by the procedure described above are all distinct because they have distinct first columns. It remains only to show that no other P -tableau whose shape is a hook with k elements in the first column can induce \mathfrak{o} . It is clear that the first column of any P -tableau T inducing \mathfrak{o} must consist of k \prec -minimal elements of \mathfrak{o} in \prec -ascending order. We claim that the only possible element that can go in the $(1, 1)$ cell of T is the \prec -minimal element u of the \prec -minimal elements of \mathfrak{o} . For suppose that the element in the $(1, 1)$ cell is some $v \neq u$. Since v is necessarily a \prec -minimal element of \mathfrak{o} , we have $u \prec v$, and therefore u cannot be in the $(1, 2)$ cell of T . We claim that u cannot actually be anywhere in the first row without violating the P -tableau condition. Suppose that w is the element in the $(1, 2)$ cell. Then w is a \prec -minimal element of the subposet \mathfrak{o}' consisting of the elements in the first row of T excluding v . So either w is a \prec -minimal element of \mathfrak{o} or w covers some element w' in the first column of T . In the former case, $u \prec w$ by definition of u , and in the latter case, u and w are \prec -comparable (since $u \in \mathfrak{o}'$ and w is \prec -minimal in \mathfrak{o}') and we

cannot have $w \prec u$ for then $w \prec u \prec u'$ would be a contradiction. Either way, $u \prec w$ so u cannot be in the $(1, 3)$ of T either. This argument can be continued inductively to show that u cannot be anywhere, a contradiction. Following a similar argument, we can show that the only possible way the elements in the first row can be arranged is according to the algorithm given previously. This completes the proof. ■

Proof of Proposition 3. Define

$$c_\ell^G \stackrel{\text{def}}{=} \sum_{\lambda: \ell(\lambda)=\ell} a_\lambda^G.$$

By Theorem 1,

$$c_\ell^G = \sum_T \text{sgn } T, \tag{3.2}$$

where the sum is over all special rim hook P -tableaux T with ℓ rim hooks. We can break up the sum (3.2):

$$c_\ell^G = \sum_\mu \sum_T \text{sgn } T, \tag{3.3}$$

where the outer sum is over all shapes μ and the inner sum is over all special rim hook P -tableaux $T = (T', T'')$ having shape μ and ℓ rim hooks. We now claim that the inner sum in (3.3) vanishes unless μ is a hook.

To prove this claim, assume that μ is not a hook, so that μ contains the cell $(2, 2)$. For any special rim hook P -tableau $T = (T', T'')$ with shape μ , let $H_1(T)$ denote the (special) rim hook of T'' containing $(1, 1)$ and let $H_2(T)$ denote the (special) rim hook of T'' containing $(2, 2)$. It is easy to see that $H_1(T) \neq H_2(T)$. Now fix any P -tableau T' of shape μ and let S be the set of all special rim hook P -tableaux having shape μ and ℓ rim hooks and having the form (T', T'') for some T'' . We now define a sign-reversing involution σ on S . This will prove the claim.

Note that if $H_2(T)$ contains some cells in the first row, say the cells $(1, m)$ through $(1, n)$ for some $m \leq n$, then all the cells $(1, 1)$ through $(1, m - 1)$ must belong to $H_1(T)$. Note also that if $H_2(T)$ does not contain any cells in the first row and if the rightmost cell of $H_2(T)$ is $(2, m)$ for some m , then all the cells $(1, 1)$ through $(1, m)$ must belong to $H_1(T)$. With this in mind we define $\sigma(T)$ as follows: $\sigma(T)$ is exactly the same as T except that $H_1(\sigma(T)) \neq H_1(T)$ and $H_2(\sigma(T)) \neq H_2(T)$. If $H_2(T)$ contains cells in the first row, then transfer these first-row cells to H_1 , i.e., let $H_2(\sigma(T))$ equal $H_2(T)$ with the first-row cells of $H_2(T)$ deleted, and let $H_1(\sigma(T))$ equal $H_1(T)$ plus the first-row cells of $H_2(T)$. If $H_2(T)$ does not contain any cells in the first row and its rightmost cell is $(2, m)$, then let $H_2(\sigma(T))$ equal $H_2(T)$ plus the first-row cells of $H_1(T)$ to the right of $(1, m)$ (including $(1, m)$ itself) and let $H_1(\sigma(T))$ equal $H_1(T)$ with all the cells to the right of $(1, m)$ (including $(1, m)$ itself) deleted. It is easy to check that the definition of σ makes sense and that it is an

involution on the set S . It is sign-reversing because the sign of H_2 is changed (its height changes by one) but the signs of all the other rim hooks remain unchanged.

Thus in (3.3) we may restrict the outer sum to hook-shapes μ . Now it is easy to see that for a given hook μ , all special rim-hook P -tableaux having shape μ and ℓ rim hooks have the same sign, namely $(-1)^{k(\mu)-\ell}$, where $k(\mu)$ is the number of cells in the first column of μ . Moreover, we claim that the number of special rim hook tabloids having shape μ and ℓ rim hooks is $\binom{k(\mu)-1}{\ell-1}$. For, because of the special condition that all rim hooks must contain at least one cell in the first column, the cells in the first row of μ must all belong to the rim hook containing $(1, 1)$, and therefore the set of special rim hook tabloids is in bijection with the set of compositions of $k(\mu)$ into ℓ parts (just look at the way the special rim hooks subdivide the set of cells in the first column).

Using the notation of Lemma 1, we may use what we know to rewrite (3.3):

$$\begin{aligned}
c_\ell^G &= \sum_{\mu} \sum_T \operatorname{sgn} T \\
&= \sum_{\mu} \sum_T (-1)^{k(\mu)-\ell} \\
&= \sum_{\mu} (-1)^{k(\mu)-\ell} \binom{k(\mu)-1}{\ell-1} \pi_\ell \\
&= \sum_k (-1)^{k-\ell} \binom{k-1}{\ell-1} \pi_\ell \\
&= \sum_k (-1)^{k-\ell} \binom{k-1}{\ell-1} \sum_m \binom{m-1}{k-1} \kappa_m \\
&= \sum_m \kappa_m \sum_k (-1)^{k-\ell} \binom{m-1}{k-1} \binom{k-1}{\ell-1} \\
&= \sum_m \kappa_m \binom{m-1}{\ell-1} \sum_k (-1)^{k-\ell} \binom{m-\ell}{k-\ell} \\
&= \sum_m \kappa_m \binom{m-1}{\ell-1} \sum_k (-1)^k \binom{m-\ell}{k} \\
&= \sum_m \kappa_m \binom{m-1}{\ell-1} \delta_{m,\ell} \\
&= \kappa_\ell.
\end{aligned}$$

This proves the proposition. ■

4. CONCLUDING REMARKS

Optimistically, Theorem 1 could be used to provide a combinatorial proof of the Poset Chain Conjecture. However, there is a caveat. In the case when P is an ordinal sum

of antichains of sizes $\nu_1, \nu_2, \dots, \nu_m$, one can show that $f_\mu^G = K_{\mu, \nu}$, the Kostka numbers. The Poset Chain Conjecture in this case amounts to the assertion that if K is the Kostka matrix, then $K^{-1}K$ is a matrix with nonnegative coefficients. While this is a trivial fact algebraically, Egecioglu and Remmel state that it is an open problem to prove this bijectively using their combinatorial interpretation. It seems therefore that Theorem 1 needs to be supplemented by algebraic arguments for it to be an effective tool for attacking the full Poset Chain Conjecture.

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