Discrete Mathematics 115 (1993) 95–102
North-Holland

Distances forbidden by two-colorings of $\mathbb{Q}^3$ and A_n

Timothy Chow
Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

Received 2 August 1990
Revised 1 April 1991

Abstract

Chow, T., Distances forbidden by two-colorings of $\mathbb{Q}^3$ and A_n, Discrete Mathematics 115 (1993) 95–102.

For $X=\mathbb{Q}^3$ or A_n (where A_n is the set of points in $\mathbb{Q}^n$ whose coordinates have odd denominators), we characterize all sets of distances $D\subset\mathbb{R}^+$ with the following property: there exists some two-coloring of X such that, for all $d\in D$, no two points in X that are a distance d apart are the same color. We also find all numbers $d_0\in\mathbb{R}^+$ such that all sets of distances $D\subset\mathbb{R}^+$ with this property retain the property under multiplication or division by d_0.

1. Introduction

A set of distances $D\subset\mathbb{R}^+$ is said to be *forbidden* by a two-coloring of $X\subset\mathbb{R}^n$ if, for every $x, y\in X$ such that $\|x-y\|\in D$, the two points x and y have different colors. When X is an additive subgroup of $\mathbb{R}^n$, we define an *odd D-cycle* in $X\subset\mathbb{R}^n$ to be a set of points $\{x_1,\dots,x_n\}\subset X$ such that $x_1+\cdots+x_n=0$, $\|x_i\|\in D$ for $i=1,\dots,n$, and n is odd. (It will sometimes be convenient to refer to such a cycle as $x_1+\cdots+x_n$ rather than as $\{x_1.\dots,x_n\}$.) In a recent paper [2], Reid et al. proved the following two results.

Proposition 1. *A set of distances $D\subset\mathbb{R}^+$ is forbidden by no two-coloring of $\mathbb{Q}^2$ iff there are $d_1, d_2\in D$ such that each of d_1 and d_2 occurs as a distance between some two points of $\mathbb{Q}^2$ and there exist $a, b\in\mathbb{Z}^+$ such that $d_1/d_2=\sqrt{a/b}$ and $a+b$ is odd.*

Proposition 2. *If $D\subset\mathbb{R}^+$ is a set of distances and if $d_0\in\mathbb{R}^+$ occurs as a distance between some two points in $\mathbb{Q}^2$, then there is an odd D-cycle in $\mathbb{Q}^2$ iff there is an odd $(d_0\cdot D)$-cycle in $\mathbb{Q}^2$, where $d_0\cdot D=\{d_0d\mid d\in D\}$.*

Correspondence to: Timothy Chow, 14 Spring Street, Apt. 1, Somerville, MA 02143, USA.

0012-365X/93/$06.00

In their paper, Reid et al. ask if the analogues of these two propositions are true for A_n and $\mathbb{Q}^3$, where

$$A_n = \{(a_1/b_1, \ldots, a_n/b_n) \mid a_i, b_i \in \mathbb{Z} \text{ and each } b_i \text{ is odd}\}.$$

Stated in its original form the question is easy – the answer is in general no, since there is no odd $\{1\}$-cycle in $\mathbb{Q}^3$ or A_n (see [1]), but $(1,0,1)+(-1,1,0)+(0,-1,-1)$ is an odd $\{\sqrt{2}\}$-cycle in $\mathbb{Q}^3$ and in A_n for $n \geqslant 3$. However, the question can be generalized to the following two problems, which we shall solve in this paper.

(1) Characterize the sets of distances that are forbidden by some two-coloring of $\mathbb{Q}^3$ (and similarly for A_n);

(2) For both $\mathbb{Q}^3$ and A_n, characterize the numbers $d \in \mathbb{R}^+$ for which there is an odd D-cycle iff there is an odd $(d \cdot D)$-cycle.

2. Distances forbidden by some two-colorings of $\mathbb{Q}^3$ and A_n

Definition. Let $X \subset \mathbb{R}^n$. We say that $d \in \mathbb{R}^+$ is *X-attainable* if there exist $x, y \in X$ such that $\|x - y\| = d$.

Remark. If $n \in \mathbb{Z}^+$ and $\sqrt{n}$ is $\mathbb{Q}^3$-attainable, then $\sqrt{n}$ is $\mathbb{Z}^3$-attainable. (This follows from the well-known fact that an integer can be written as the sum of three integer squares iff it is not of the form $4^m(8r+7)$, where m and r are nonnegative integers [3]. For if $n = (a_1/b)^2 + (a_2/b)^2 + (a_3/b)^2$, then nb^2 is not of the form $4^m(8r+7)$, and therefore neither is n, since b^2, being the square of an integer, is of the form $4^m(8r+1)$.)

Definition. If $X \subset \mathbb{Q}^n$ and $d \in \mathbb{R}^+$ is X-attainable, then d can be written in the form $\sqrt{2^k p/q}$ with p and q odd positive integers and k an integer that is uniquely determined by d. We say that k is the *two-index* of d.

Remark. The two-index of an A_n-attainable distance is nonnegative.

Our main result is the following.

Theorem 1. *Let $D \subset \mathbb{R}^+$, let $X \in \{\mathbb{Q}^3, A_1, A_2, A_3, \ldots\}$ and let D' be the set of distances in D that are X-attainable. If D is forbidden by some two-coloring of X, then every element of D' has the same two-index k. Moreover,*

(a) *if $X = \mathbb{Q}^3$, then k is even;*

(b) *if $X = A_n$ and $n \geqslant 4$, then $k = 0$;*

(c) *if $X = A_3$ or A_1, then k is even and nonnegative; and*

(d) *if $X = A_2$, then $k \geqslant 0$.*

Conversely, these necessary conditions on D' are sufficient to ensure that D is forbidden by some two-coloring of X.

Remark. For $n>3$, the only distances that are forbidden by two-colorings of $\mathbb{Q}^n$ are those that are not $\mathbb{Q}^n$-attainable. (For it is easy to construct an odd $\{1\}$-cycle, and by using a matrix similar to the one given below in Case 2 of the proof of Theorem 1, we can obtain an odd $\{d\}$-cycle for any $\mathbb{Q}^n$-attainable distance d.) Thus Theorem 1 completely solves the problem of distances forbidden by two-colorings of $\mathbb{Q}^n$ and A_n for all n.

We need a few results from [1, 2]. The first proposition below is a special case of the well-known result that a graph is bipartite if and only if it does not contain an odd cycle.

Proposition 3. *Let* $X\subset\mathbb{R}^n$ *and* $D\subset\mathbb{R}^+$. *Then D is forbidden by no two-coloring of X iff there is an odd D-cycle in X.*

Proposition 4 (Johnson). *There exist two-colorings of* $\mathbb{Q}^3$ *and* A_n *that simultaneously forbid all distances of the form* $\sqrt{p/q}$ *with p and q odd positive integers.*

Proposition 5 (Reid et al.). *Let* $d=\sqrt{p/q}$ *with p even and q odd. If d is* A_n*-attainable, then no two coloring of* A_n *forbids* $\{1,d\}$.

To prove Theorem 1 we also need the following lemmas.

Lemma 1. *If* $n\equiv 2 \bmod 4$, *then there is an odd* $\{\sqrt{n}\}$*-cycle in* $\mathbb{Z}^3$.

Proof. Suppose $n\equiv 2 \bmod 4$. Then n is not of the form $4^m(8r+7)$, so we can write $n=a^2+b^2+c^2$ for some $a,b,c\in\mathbb{Z}$. Since $n\equiv 2 \bmod 4$, exactly two of a, b and c are odd; without loss of generality we can assume that a and b are odd and c is even. If $a-b+c=0$, then

$$(a,-b,c)+(c,a,-b)+(-b,c,a)=(a-b+c,a-b+c,a-b+c)=(0,0,0)$$

is an odd $\{\sqrt{n}\}$-cycle in $\mathbb{Z}^3$ and we are done. Otherwise, write $a+b+c=2^{k_1}p_1$ and $a-b+c=2^{k_2}p_2$ with p_1 and p_2 odd. Since $n=a^2+b^2+c^2$ is even, $a+b+c$ and $a-b+c$ are even, and since b is odd, exactly one of $a+b+c$ and $a-b+c$ is congruent to 2 mod 4, so that $k_1\neq k_2$; without loss of generality we can assume $k_1>k_2$. Then

$$\begin{aligned}
&2^{k_1-k_2}p_1(a,-b,c)+2^{k_1-k_2}p_1(c,a,-b)+2^{k_1-k_2}p_1(-b,c,a)-p_2(a,b,c)\\
&\quad -p_2(c,a,b)-p_2(b,c,a)\\
&\qquad =2^{k_1-k_2}p_1(a-b+c,a-b+c,a-b+c)-p_2(a+b+c,a+b+c,a+b+c)\\
&\qquad =2^{k_1-k_2}p_1(2^{k_2}p_2,2^{k_2}p_2,2^{k_2}p_2)-p_2(2^{k_1}p_1,2^{k_1}p_1,2^{k_1}p_1)\\
&\qquad =(0,0,0),
\end{aligned}$$

and since

$$2^{k_1-k_2}p_1+2^{k_1-k_2}p_1+2^{k_1-k_2}p_1+p_2+p_2+p_2=3(2^{k_1-k_2}p_1+p_2)$$

is odd, this gives us an odd $\{\sqrt{n}\}$-cycle in $\mathbb{Z}^3$. □

Lemma 2. *Let n_1, n_2 and k be positive integers with n_1 and n_2 odd. If $\sqrt{n_1}$ and $\sqrt{n_2}$ are $\mathbb{Z}^3$-attainable, then there is an odd $\{\sqrt{n_1},\ 2^k\sqrt{n_2}\}$-cycle in $\mathbb{Z}^3$.*

Proof. Assume that $\sqrt{n_1}$ and $\sqrt{n_2}$ are $\mathbb{Z}^3$-attainable. Write $n_1=a_1^2+b_1^2+c_1^2$ and $n_2=a_2^2+b_2^2+c_2^2$ with $a_1,b_1,c_1,a_2,b_2,c_2\in\mathbb{Z}$. Let $p_1=a_1+b_1+c_1$ and let $p_2=a_2+b_2+c_2$. Since n_1 and n_2 are odd, p_1 and p_2 are odd. Now a simple calculation (cf. Lemma 1) shows that

$$\begin{aligned}&p_1(2^ka_2,2^kb_2,2^kc_2)+p_1(2^kc_2,2^ka_2,2^kb_2)+p_1(2^kb_2,2^kc_2,2^ka_2)\\&\quad-2^kp_2(a_1,b_1,c_1)-2^kp_2(c_1,a_1,b_1)-2^kp_2(b_1,c_1,a_1)\\&\quad=(0,0,0),\end{aligned}$$

and since

$$p_1+p_1+p_1+2^kp_2+2^kp_2+2^kp_2=3(p_1+2^kp_2)$$

is odd, this gives us an odd $\{\sqrt{n_1},2^k\ \sqrt{n_2}\}$-cycle in $\mathbb{Z}^3$. □

Lemma 3. *Let $X=\mathbb{Q}^3$ or A_n for $n\geqslant 3$, and let $D\subset\mathbb{R}^+$ be a set of X-attainable distances. If D is forbidden by some two-coloring of X, then every $d\in D$ has even two-index.*

Proof. Assume that D is forbidden by some two-coloring of X. By Proposition 3, there are no odd D-cycles in X. Let $d\in D$ and write $d=\sqrt{2^kp/q}$ with p and q odd positive integers. Suppose k is odd; write $k=2m+1$ so that $d=2^m\sqrt{2p/q}$. By Lemma 1, there is an odd $\{\sqrt{2pq}\}$-cycle in $\mathbb{Z}^3\subset X$. We may scale this cycle by a factor of $2^m/q$ to obtain an odd $\{d\}$-cycle in X (since in the case $X=A_n$, $m\geqslant 0$ so that the denominator of $2^m/q$ is odd). But this contradicts the non-existence of odd D-cycles in X, so k is even and the result follows. □

Now we are ready for the proof of our main result.

Proof of Theorem 1. There are five cases, which we shall consider separately.

Case 1: $X=\mathbb{Q}^3$.

To show that the conditions are sufficient, we can (by Proposition 4) find a two-coloring of X that forbids all distances of the form $\sqrt{p/q}$ with p and q odd, and then we can scale this two-coloring by a factor of $2^{k/2}$ to obtain a coloring that forbids D' and hence D. To show necessity, assume that D is forbidden by some two-coloring of X.

Lemma 3 implies that every element of D' has even two-index. It remains to show that any two elements $d_1, d_2 \in D'$ have the same two-index. Write $d_1 = 2^{k_1}\sqrt{p_1/q_1}$, $d_2 = 2^{k_2}\sqrt{p_2/q_2}$, where $k_1, p_1, q_1, k_2, p_2, q_2 \in \mathbb{Z}$ and p_1, q_1, p_2, q_2 are odd. Suppose $k_1 \neq k_2$; without loss of generality we may assume that $k_1 > k_2$. Since d_1 and d_2 are X-attainable, $\sqrt{p_1 q_1 q_2^2}$ and $\sqrt{p_2 q_2 q_1^2}$ are $\mathbb{Q}^3$-attainable and therefore $\mathbb{Z}^3$-attainable. Hence we can apply Lemma 2 to find an odd $\{2^{k_1-k_2}\sqrt{p_1 q_1 q_2^2}, \sqrt{p_2 q_2 q_1^2}\}$-cycle in $\mathbb{Z}^3 \subset X$; if we scale this cycle by a factor of $2^{k_2}/q_1 q_2$, we obtain an odd $\{d_1, d_2\}$-cycle in X. But Proposition 3 implies that there are no odd D'-cycles in X. This contradiction establishes that $k_1 = k_2$ as required.

Case 2: $X = A_n$ *with* $n \geqslant 4$.

Sufficiency follows from Proposition 4. To show necessity, it is enough to show (in view of Lemma 3 and Proposition 3) that if $d_0 = 2^k\sqrt{p/q}$ for some odd positive integers p and q and some $k \geqslant 1$, then there is an odd $\{d_0\}$-cycle in A_4 (since every higher A_n contains an isomorph of A_4). Given such a d_0, we can write $pq = a^2 + b^2 + c^2 + d^2$ for some $a, b, c, d \in \mathbb{Z}$ (since every positive integer is the sum of four squares [3]). Now let

$$C = \{(2,0,0,0), (1,1,1,1), (-1,1,-1,-1), (-1,-1,1,-1), (-1,-1,-1,1)\}.$$

Then C is an odd $\{2\}$-cycle in A_4. Let T be the linear transformation given by the matrix

$$\frac{2^{k-1}}{q}\begin{pmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{pmatrix}.$$

It is easy to check that $T(C)$ is an odd $\{2^k\sqrt{p/q}\}$-cycle, and since $k \geqslant 1$, $2^{k-1}/q$ has odd denominator, so $T(C) \subset A_4$ and we are done.

Case 3: $X = A_3$.

To prove necessity, we can just take the argument given in Case 1 and replace X with A_3 everywhere, provided we observe that in this case $k_2 \geqslant 0$ so that the denominator of $2^{k_2}/q_1 q_2$ is odd, and hence an odd D'-cycle in A_3 remains in A_3 upon multiplication by $2^{k_2}/q_1 q_2$. To prove sufficiency, suppose D satisfies the necessary conditions. Some two-coloring of $\mathbb{Q}^3$ forbids D, so a fortiori some two-coloring of A_3 forbids D.

Case 4: $X = A_2$.

Sufficiency follows from Proposition 1: any set D satisfying the given conditions is forbidden by some two-coloring of $\mathbb{Q}^2$ (and thus of $X \subset \mathbb{Q}^2$). To show necessity, assume that D is forbidden by some two-coloring of A_2, so that by Proposition 3 there are no odd D'-cycles in A_2. We need only show that any two elements $d_1, d_2 \in D'$ must have the same two-index. Write $d_1 = \sqrt{2^{k_1} p_1/q_1}$ and $d_2 = \sqrt{2^{k_2} p_2/q_2}$, with p_1, q_1, p_2, q_2 odd positive integers. Assume without loss of generality that $k_1 > k_2$. The product of a sum of two squares and a sum of two squares is a sum of two squares, so $d_1 d_2$ is $\mathbb{Q}^2$-attainable and therefore so is

$$\left(\frac{p_2 q_1}{d_2^2}\right) d_1 d_2 = \sqrt{2^{k_1-k_2} p_1 p_2 q_1 q_2}.$$

Let $a = 2^{k_1-k_2} p_1 p_2 q_1 q_2$; then a is the sum of two rational squares, and since $k_1 > k_2$, a is an integer. Hence a is the sum of two integer squares. We see from the relation $\sqrt{a}/p_2 q_1 = d_1/d_2$ that d_1/d_2 is A_2-attainable, with nonzero two-index. Thus by Proposition 3 and Proposition 5, there is an odd $\{1, d_1/d_2\}$-cycle in A_2. Let $(x, y) \in A_2$ be a distance d_2 from (0, 0). Transform the odd $\{1, d_1/d_2\}$-cycle with the matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

to obtain an odd $\{d_1, d_2\}$-cycle in A_2. This contradicts the non-existence of odd D'-cycles in A_2 and the result follows.

Case 5: $X = A_1$.

Sufficiency follows from Proposition 1 as in Case 4. To show necessity, suppose that D is forbidden by some two-coloring of A_1, so that by Proposition 3 there are no odd D'-cycles in A_1. That the two-index of every element of D' is even and nonnegative is clear; suppose p_1/q_1 and p_2/q_2 are elements of D' where q_1 and q_2 are odd and p_1/q_1 has a higher two-index than p_2/q_2. Then by Proposition 3 and Proposition 5, there is an odd $\{1, p_2 q_1/p_1 q_2\}$-cycle in A_1, which we can multiply by p_1/q_1 to give an odd $\{p_1/q_1, p_2/q_2\}$-cycle in A_1, contradicting the non-existence of odd D'-cycles in A_1. This completes the proof. □

3. Legal scalefactors

Definition. Let $X \in \{\mathbb{Q}^3, A_1, A_2, A_3, \ldots\}$. We say that $d \in \mathbb{R}^+$ is a *legal scalefactor* for X if, for all $D \subset \mathbb{R}^+$, there is an odd D-cycle in X iff there is an odd $(d \cdot D)$-cycle in X.

Lemma 4. *If $d \in \mathbb{R}^+$ is not X-attainable, then d is not a legal scalefactor.*

Proof. If there is no odd $\{2d\}$-cycle in X, then simply observe that there is an odd $\{1, 2\}$-cycle in X but there is no odd $\{d, 2d\}$-cycle in X. Otherwise, $2d$ must be X-attainable, but d is not, so X must be A_n for some $n \geq 3$, and moreover $2d$ must have two-index zero or one. (This follows because the A_3-attainable numbers are precisely the numbers of the form $\sqrt{p}/q$ with q odd and p not of the form $4^m(8r+7)$, and for $n \geq 4$ the A_n-attainable numbers are precisely the numbers of the form $\sqrt{p}/q$ with q odd.) But there is an odd $\{2d\}$-cycle in X, so by Proposition 3 and Proposition 4 the two-index of $2d$ cannot be zero and must be one. Now observe that there is an odd

$\{\sqrt{2}\}$-cycle in X but no odd $\{d\sqrt{2}\}$-cycle in X (since the two-index of $d\sqrt{2}$ is zero), so d cannot be a legal scale factor. □

Theorem 2. *If $d \in \mathbb{R}^+$ is a legal scalefactor for A_n, then d is A_n-attainable with two-index zero. If $n \neq 3$, then the converse also holds.*

Proof. ($\Rightarrow$) By Lemma 4, d is A_n-attainable. Write $d = \sqrt{2^k p/q}$ with p and q odd positive integers and $k \geqslant 0$. Suppose $k > 0$. Then by Proposition 3 and Proposition 5, there is an odd $\{1, d\}$-cycle in A_n. Since $1/d$ has negative two-index, it is not A_n-attainable, so by Theorem 1 and Proposition 3 there is no odd $\{1, 1/d\}$-cycle in A_n. Hence d is not a legal scalefactor.

($\Leftarrow$) This follows from Theorem 1 and the fact that for $n \neq 3$, the product of two A_n-attainable numbers is A_n-attainable. □

Theorem 3. *Let $d \in \mathbb{R}^+$. Then d is a legal scalefactor for $\mathbb{Q}^3$ if we can write $d = r\sqrt{n}$ with $r \in \mathbb{Q}$ and $n \equiv 1 \bmod 8$.*

Proof. We shall freely use the theorem about expressing integers as the sum of three squares that was mentioned earlier.

($\Rightarrow$) Assume that d is a legal scalefactor. Then by Lemma 4 there is a rational multiple d' of d that can be written in the form $d' = \sqrt{n}$ with n squarefree. Clearly d' is a legal scalefactor for $\mathbb{Q}^3$; it remains to show that $n \equiv 1 \bmod 8$. Theorem 1 implies that n must be odd. Since d' is $\mathbb{Q}^3$-attainable, $n \not\equiv 7 \bmod 8$. Now suppose $n \equiv 3 \bmod 8$. Then by Lemma 2, there is an odd $\{\sqrt{n}, 2\sqrt{5}\}$-cycle in $\mathbb{Q}^3$, but there is no odd $\{n, 2\sqrt{5n}\}$-cycle in $\mathbb{Q}^3$ (since $5n \equiv 7 \bmod 8$ and thus $2\sqrt{5n}$ is not $\mathbb{Q}^3$-attainable), contradicting the fact that $\sqrt{n}$ is a legal scalefactor for $\mathbb{Q}^3$. So $n \not\equiv 3 \bmod 8$. Similarly, $n \not\equiv 5 \bmod 8$, for then there would be an odd $\{\sqrt{n}, 2\sqrt{3}\}$-cycle in $\mathbb{Q}^3$ but no odd $\{n, 2\sqrt{3n}\}$-cycle in $\mathbb{Q}^3$.

($\Leftarrow$) Since $n \equiv 1 \bmod 8$, any $d_0 \in \mathbb{R}^+$ is $\mathbb{Q}^3$-attainable iff $d_0 d$ is; now apply Theorem 1. □

Theorem 4. *Let $d \in \mathbb{R}^+$. Then d is a legal scalefactor for A_3 iff we can write $d = p\sqrt{n}/q$ with p and q odd integers and $n \equiv 1 \bmod 8$.*

Proof. If d is a legal scalefactor, then Lemma 4 and Theorem 2 together imply that we can write $d = p\sqrt{n}/q$ for some odd positive integers p, q and n; the remainder of the proof is the same as that of Theorem 3 except with A_3 in place of $\mathbb{Q}^3$. □

Acknowledgment

I would like to thank David Moulton for many valuable discussions and for some ideas that have been incorporated into the proof of Lemma 1.

This work was done under the supervision of Dr. Joseph A. Gallian at the University of Minnesota, Duluth with the financial support of the National Science Foundation (grant number DMS 9000742) and the National Security Agency (grant number MDA 904-88-H-2027).

References

[1] P.D. Johnson Jr, Two-colorings of a dense subgroup of $\mathbb{Q}^n$ that forbid many distances, Discrete Math. 79(2) (1989/90) 191–195.

[2] M. Reid, D. Jungreis and D. Witte, Distances forbidden by some two-colorings of $\mathbb{Q}^2$, Discrete Math. 82 (1990) 53–56.

[3] J.P. Serre, A Course in Arithmetic, Graduate Texts in Mathematics (Springer, New York, 1973) appendix to Chapter IV.