Important note (3/22/97). This is a paper that I wrote in 1993 that is unpublishable in its current form and will probably never be published in any form. Nevertheless, it has been cited by other people (R. P. Stanley, Research problem 251: Spanning trees of Aztec diamonds, *Discrete Math.* **157** (1996), 383–385; and C. Athanasiadis's Ph.D. thesis, M.I.T., 1996) so I am making it available to those who are interested.

Why is the paper unpublishable? After I wrote the paper, it was pointed out to me that an alternating strip and its conjugate are just the two connected components of the tensor product of a path and a bipartite graph. The ordinary spectrum of a graph behaves well under tensor product—the eigenvalues of the tensor product are just the pairwise products of the eigenvalues of the factor graphs—and from this it is not difficult to derive Theorem 1. (Chebyshev polynomials arise from computing the spectrum of a path.)

Conjecture 2 is now a theorem, proved first by Donald Knuth ("Aztec diamonds, checkerboard graphs, and spanning trees," to appear in *J. Alg. Combin.*). In that paper Knuth also computes the spectra of the even and odd Aztec diamonds, thereby proving Conjecture 1.

This should explain why the paper is unpublishable. However, there are a few things in it that are still interesting. The "standard" approach to proving Theorem 1 is algebraic; in contrast, the proof here is combinatorial: a fact about eigenvalues is proved by counting walks. The Ph.D. thesis of Christos Athanasiadis (M.I.T., 1996) gives some more examples of this paradigm.

Incidentally, once it is observed that G_n^1 and G_n^2 are the two connected components of a tensor product, it is practically trivial to establish a bijection between closed walks on G_n^1 and G_n^2. For every closed walk in a tensor product graph "factors" into a "product" of closed walks in its factor graphs. In a bipartite graph, the closed walks alternate between "black" and "white" vertices. There is a bijection between closed walks that start on black vertices and closed walks that start on white vertices. This bijection gives rise to the desired bijection between closed walks on G_n^1 and G_n^2. This idea extends easily to Godsil and McKay's "partitioned tensor products," giving easy combinatorial proofs of most of the results in their paper "Products of graphs and their spectra" (in *Combinatorial Mathematics IV,* eds. L. R. A. Casse and W. D. Wallis, Lecture Notes in Mathematics #560, Springer, Berlin, 1976, pp. 61–72). I haven't bothered publishing this either because Brendan McKay told me that people aren't much interested in partitioned tensor products, but then again Donald Knuth seems to have gotten interested in them recently...

Finally, although there now exist three different proofs of Stanley's "factor-of-4" conjecture, the proof of Theorem 2 below is still rather interesting. It is a fairly straightforward transfer-matrix argument, but with a slight twist that may prove helpful in other contexts.

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Spectra and Complexity of Periodic Strips

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Abstract. The graph whose vertices are the squares of a $(2n+1) \times (2n+1)$ chessboard and whose edges connect precisely those pairs of squares that are diagonally adjacent has two connected components. The component that includes the central square is called the odd Aztec diamond of order n (or OD_n) and the other component is called the even Aztec diamond of order n (or AD_n). We prove a conjecture of Stanley that the spectra of OD_n and AD_n are identical up to the multiplicity of zero; in fact we compute the spectra explicitly. We also give a partial result towards another conjecture of Stanley: that AD_n has four times as many spanning trees as OD_n. Our results generalize readily to any periodic strip—i.e., a one-way infinite strip such that some finite translation is an isomorphism into itself.

1. Introduction

The graph whose vertices are the squares of a $(2n+1) \times (2n+1)$ chessboard and whose edges connect precisely those pairs of squares that are diagonally adjacent has two connected components. The component that includes the central square is the *odd Aztec diamond of order n* (denoted by OD_n) and the other component is the *even Aztec diamond of order n* (denoted by AD_n). Aztec diamonds were first introduced in [3], where it is shown that the number of perfect matchings of the even Aztec diamond of order n is $2^{n(n+1)/2}$. (Warning: our notation differs slightly from that of [3]; if we connect *orthogonally* adjacent squares of their Aztec diamond of order n then the resulting graph is isomorphic to what we have called the even Aztec diamond of order n.) This remarkable result suggests that it might be fruitful to investigate other invariants of Aztec diamonds. The starting point for this paper is the following pair of conjectures by Stanley. (Recall that the *complexity $\kappa(G)$* of a graph G is the number of spanning trees of G.)

Conjecture 1. *For every n, the spectra of OD_n and AD_n are identical except for the multiplicity of zero.*

Conjecture 2. *For every n, $\kappa(\mathrm{AD}_n)/\kappa(\mathrm{OD}_n) = 4$.*

In this paper we prove a generalization of Conjecture 1 and compute the spectra of the Aztec diamonds explicitly. We also give a partial result in the direction of Conjecture 2, although Conjecture 2 itself remains open.

2. Periodic strips

The proofs of our main results apply not only to Aztec diamonds but also to a slightly more general class of graphs which we shall now describe precisely. If G is a graph and S is a subset of the vertices of G, we write $G(S)$ for the subgraph of G induced by S.

Now for our main definition.

Definition. A *periodic strip* is an infinite undirected graph G together with

  a. an ordered partition (S_k) of its vertices into finite nonempty sets (the *blocks* of G),

  b. an integer p (the *period* of G), and

  c. a sequence of bijections $(\varphi_k : S_k \to S_{k+p})$

such that

  1. every edge connects a vertex in S_k with a vertex in S_{k+1} for some k; and

  2. for every k, the map

$$\varphi_k \oplus \varphi_{k+1} : S_k \oplus S_{k+1} \to S_{k+p} \oplus S_{k+p+1}$$

    induces an isomorphism from $G(S_k \oplus S_{k+1})$ onto $G(S_{k+p} \oplus S_{k+p+1})$.

(By abuse of notation we sometimes write $\varphi_k \oplus \varphi_{k+1}$ for the *graph* map as well as for the *set* map.) See Figure 1(a) for an example.

Given such a periodic strip G, we define

$$G_n \stackrel{\text{def}}{=} G(S_0 \cup S_1 \cup \cdots \cup S_{pn})$$

for all $n > 0$. The *complexity sequence* of G is the sequence of numbers

$$\kappa(G_1), \kappa(G_2), \kappa(G_3), \ldots .$$

For $1 \le k \le p-1$ we define the *kth conjugate* G^k to be the periodic strip obtained from G by deleting the first $k-1$ blocks (so that the blocks of G^k are S_k, S_{k+1}, etc.). Note that $G^1 = G$.

In the special case of $p = 2$ we say that G is an *alternating strip* if it possesses the following additional property: for every $k \ge 0$ the map

$$\varphi_k \oplus I_{k+1} : G(S_k \oplus S_{k+1}) \to G(S_{k+2} \oplus S_{k+1})$$

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(where I_{k+1} is the identity map on S_{k+1}) is a graph isomorphism. Intuitively, G flips back and forth between some bipartite graph and its mirror image. See Figure 1(b) for an example.

[Include Figure 1 here]

We can now state our main theorems.

Theorem 1. *If G is an alternating strip then for all n the spectra of G_n^1 and G_n^2 are identical except possibly for the multiplicity of zero.*

Theorem 2. *If G is a periodic strip then the complexity sequence of G satisfies a linear recurrence with constant coefficients. Moreover, the complexity sequence of any conjugate of G satisfies exactly the same recurrence.*

We remark that Theorem 1 can be interpreted as saying that the conjugates of alternating strips are "almost isospectral." Many classes of isospectral graphs are known (see for example section 6.1 of [2]), but Theorem 1 appears to be new.

To see the relation between the above theorems and the Aztec diamond conjectures, consider the chessboard of width $2n + 1$ and infinite length with the diagonal adjacency relation mentioned in the introduction. If $O^{(n)}$ and $A^{(n)}$ are the two connected components, then it is easy to see that $O^{(n)}$ and $A^{(n)}$ are alternating strips and in fact are (isomorphic to) conjugates of each other. Moreover, $O_n^{(n)} = \mathrm{OD}_n$ and $A_n^{(n)} = \mathrm{AD}_n$. Thus Theorem 1 implies Conjecture 1.

As for Conjecture 2, the first thing that our results suggest is that we should generalize and conjecture that $\kappa(A_i^{(n)})/\kappa(O_i^{(n)}) = 4$ for *all* i, not just for $i = n$. For Theorem 2 implies that to check this generalized conjecture for any fixed n, we need only verify it for finitely many i, and then the recurrence will take care of the rest. By direct computation we have verified that this generalized conjecture is valid for $n \leq 3$, but of course this does not resolve Conjecture 2.

3. Spectra

We shall give two proofs of Theorem 1. The first proof has the advantage of being more combinatorial and the second proof has the advantage of showing how to compute the spectrum explicitly. For our first proof we need just one preliminary result, which we shall now state.

Given a finite graph H, let A_H be its adjacency matrix, let $q_H(\lambda) = \det(I - \lambda A_H)$, and let $c_H(s)$ be the number of closed walks of length s in H. (For our purposes, cyclic shifts of

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closed walks are thought of as being distinct.) The following proposition is Corollary 4.7.3 in [4].

Proposition 1. *For any finite graph H,*

$$\sum_{s \geq 1} c_H(s) \lambda^s = -\frac{\lambda q'_H(\lambda)}{q_H(\lambda)}. \quad \square$$

Now for our first proof.

First proof of Theorem 1. We shall denote the blocks and bijections of G by S_k and φ_k respectively, with $k \geq 0$. Fix any integer $n > 0$. We claim that it suffices to prove that

$$c_{G_n^1}(2m) = c_{G_n^2}(2m) \tag{3.1}$$

for all integers $m > 0$. To see this, note first that since G_n^1 and G_n^2 are bipartite, there are no closed walks of odd length in either graph. Thus if we can prove (3.1) it will follow from Proposition 1 that

$$-\frac{\lambda q'_{G_n^1}(\lambda)}{q_{G_n^1}(\lambda)} = -\frac{\lambda q'_{G_n^2}(\lambda)}{q_{G_n^2}(\lambda)}.$$

It is easy to see that this implies that

$$q_{G_n^1}(\lambda) = r q_{G_n^2}(\lambda)$$

for some constant r (for example, by cancelling the λ's and integrating both sides with respect to λ). But setting $\lambda = 0$ in the definition of $q_H(\lambda)$ shows that $q_H(0) = 1$ for any graph H, so $r = 1$. Since for any graph H the roots of $q_H(\lambda)$ are just the reciprocals of the nonzero eigenvalues of H with the correct multiplicities, it follows at once that the spectra of G_n^1 and G_n^2 are identical except possibly for the multiplicity of zero, as desired.

To prove (3.1), fix any integer $m > 0$. We now describe a bijection β between the set of closed walks of length $2m$ in G_n^1 and the set of closed walks of length $2m$ in G_n^2. Let

$$W = (u_0, u_1, \ldots, u_{2m-1})$$

be a closed walk of length $2m$ in G_n^1, where the u_i are the vertices traversed. (We understand the subscripts to be labelled modulo $2m$, i.e., $u_{2m} = u_0$, etc.) Let k_i be the block number of u_i, so that $u_i \in S_{k_i}$. Set

$$\beta(W) = (v_0, v_1, \ldots, v_{2m-1})$$

where the v_i are defined by

$$v_i = \begin{cases} u_i, & \text{if } k_i = k_{i+1} + 1; \\ \varphi_{k_i}(u_i), & \text{if } k_i = k_{i+1} - 1. \end{cases}$$

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(By property 1 of periodic strips the two cases in this definition exhaust the possibilities.)

First let us verify that $\beta(W)$ is indeed a closed walk in G_n^2. Clearly $v_{2m} = v_0$ so we just need to show that v_{i-1} and v_i are adjacent for $i \in \{1, 2, \ldots, 2m\}$. Consider first the case where $k_{i-1} = k_i + 1$, so that $v_{i-1} = u_{i-1}$. If $v_i = u_i$ then v_{i-1} and v_i are adjacent because u_{i-1} and u_i are. Otherwise $v_i = \varphi_{k_i}(u_i)$. Since G is an alternating strip,

$$\varphi_{k_i} \oplus I_{k_{i-1}} = \varphi_{k_i} \oplus I_{k_i+1}$$

is an isomorphism, and since it maps u_i to v_i and u_{i-1} to v_{i-1}, v_{i-1} and v_i must be adjacent, as required. Similar reasoning applies to the other case, where $k_{i-1} = k_i - 1$.

To see that $\beta(W)$ lies entirely within G_n^2, simply note that the block number of v_i is $k_{i+1} + 1$.

To see that β is bijective, we actually need only show that β is injective, i.e., that the number of closed walks in G_n^1 does not exceed the number of closed walks in G_n^2, because if we can show this then we can use the same argument with G^2 in place of G and since $G^3 \cong G$, this proves the inequality in the other direction.

To recover W from $\beta(W)$, observe as before that k_i is one less than the block number of v_{i-1}, so that the k_i are easily recovered. Now u_i is either v_i or the inverse image of v_i under some φ_k, and the choice between these options depends only on the block sequence, so the u_i are also easily recovered. $\square$

For our second proof a few more preliminaries are necessary. We fix the following notation for the remainder of this section. For $n \geq 1$, let A_n be the following $n \times n$ matrix

$$A_n = \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$$

and let P_n be the graph with adjacency matrix A_n.

The next definition is so natural that it is probably not new, but for lack of a good reference we include full details here.

Definition. Given a connected bipartite graph H and one of its parts V, the *walk graph* $W[H, V]$ is the multidigraph on the vertex set V such that the number of edges from u to v is the number of walks of length two from u to v in H.

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Lemma 1. *Let H be a connected bipartite graph and let V be one of its parts. Then the spectrum of $W[H, V]$ is nonnegative and the nonzero spectrum of H is given by the multiset $\{\pm\sqrt{w}\}$ where w runs through the nonzero spectrum of $W[H, V]$.*

Proof. Recall that for our purposes, cyclic shifts of closed walks are considered to be distinct. It follows that the number of closed walks of length $2m$ in H is twice the number of closed walks of length m in $W[H, V]$. Now the number of closed walks of length m is just the sum of the mth powers of the eigenvalues (see for example the proof of Corollary 4.7.3 in [4]), so if we denote the nonzero eigenvalues of H by λ_i and the nonzero eigenvalues of $W[H, V]$ by μ_i we have

$$\sum_i \lambda_i^{2m} = 2 \sum_i \mu_i^m \tag{3.2}$$

for all $m > 0$. It is well-known (see for example Theorem 3.3 of [2]) that if λ is an eigenvalue of a bipartite graph then so is $-\lambda$, and moreover the multiplicity is the same. Therefore if we take only the positive λ_i in (3.2) we obtain

$$\sum_i (\lambda_i^2)^m = \sum_i \mu_i^m.$$

Now it is again well-known (and easy to prove, e.g., by using Vandermonde determinants) that if the sum of the mth powers of two multisets of nonzero complex numbers is the same for all m then the multisets must be equal. Therefore the multiset of the nonzero μ_i coincides with the multiset of the squares of the positive λ_i and the result follows. $\quad\square$

In particular we remark that if (V_1, V_2) is the bipartition of H, then $W[H, V_1]$ and $W[H, V_2]$ have the same nonzero spectrum even if V_1 and V_2 are not the same size.

We are now ready to give our second proof.

Second proof of Theorem 1. Again we denote the blocks of G by S_k with $k \geq 0$. Observe that G_n^1 is bipartite; let V_n^1 be the part containing S_1. Similarly let V_n^2 be the part of G_n^2 containing S_2. By Lemma 1 it suffices to show that the spectra of $W[G_n^1, V_n^1]$ and $W[G_n^2, V_n^2]$ are equal except possibly for the multiplicity of zero.

The key observation is that

$$W[G_n^1, V_n^1] \cong P_n \times W[G(S_1 \cup S_2), S_1] \qquad \text{and}$$
$$W[G_n^2, V_n^2] \cong P_n \times W[G(S_1 \cup S_2), S_2],$$

where $\times$ denotes graph product (i.e., the adjacency matrix of the product is the tensor product of the adjacency matrices of the multiplicands). To see this, take any two vertices u and v in V_n^1. If u and v lie in the same S_k, then every walk of length two from u to v must proceed via either S_{k-1} or S_{k+1}, and moreover since G is an alternating strip there

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is a natural bijection between the walks that proceed via S_{k-1} and the walks that proceed via S_{k+1}. Because G is a periodic strip we see that there is also a natural bijection between the walks that proceed via S_{k+1} and the edges of $W[G(S_1 \cup S_2), S_1]$. The other possibility is that u is in S_k and v is in S_{k+2} for some k; in this case every walk of length two must pass through S_{k+1}. Again because G is an alternating strip, such walks are in bijection with the edges of $W[G(S_1 \cup S_2), S_1]$. It is now easy to see from the definition of a product graph that $W[G_n^1, V_n^1]$ and $W[G_n^2, V_n^2]$ factor as claimed. See Figure 2 for an illustration.

[Include Figure 2 here]

Now the eigenvalues of a product graph are given by $\{\lambda_i \mu_j\}$ as λ_i and μ_j run independently through the spectra of the multiplicands. (See Theorem 2.23 of [2].) By the remark immediately following Lemma 1 we are done. $\square$

Our second proof suggests how to compute the spectrum of the Aztec diamonds explicitly. As a preliminary step we compute the spectrum of A_n. It turns out that the characteristic polynomial of A_n is essentially a Chebyshev polynomial. This should not be too surprising since the characteristic polynomial of a path is known to be a Chebyshev polynomial (see section 2.6 of [2]). The precise statement appears in the following lemma. (Note: our notation for Chebyshev polynomials follows that of [1].)

Lemma 2. *For $n \geq 1$, let $p_n(t) = \det(tI - A_n)$. Then*

$$t p_n(t^2) = S_{2n+1}(t), \tag{3.3}$$

where S_n is a Chebyshev polynomial of the first kind, and hence the eigenvalues of A_n are

$$\left\{ 4 \cos^2 \frac{k\pi}{2n+2} \;\middle|\; 1 \leq k \leq n \right\}.$$

Proof. From (3.3) one can deduce the eigenvalues of A_n immediately because the roots of S_n are

$$\left\{ 2 \cos \frac{k\pi}{n+1} \;\middle|\; 1 \leq k \leq n \right\}.$$

(See 22.3.16 and 22.5.13 of [1].) To establish (3.3), set $p_0 = 1$. By expanding along the first row we obtain the recurrence

$$p_n(t) = (t-2)p_{n-1}(t) - p_{n-2}(t)$$

for $n \geq 2$. From this we easily obtain the generating function

$$\sum_{n \geq 0} t p_n(t^2) z^n = \frac{t}{1 + (2 - t^2)z + z^2}.$$

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On the other hand 22.9.10 of [1] states that

$$\sum_{n \geq 0} U_n(t) \, z^n = \frac{1}{1 - 2tz + z^2},$$

where $U_n(t) = S_n(2t)$ is a Chebyshev polynomial of the second kind. By changing z to $-z$ and subtracting, and also changing t to $t/2$, we obtain

$$\sum_{n \geq 0} S_{2n+1}(t) \, z^{2n+1} = \sum_{n \geq 0} U_{2n+1}(t/2) \, z^{2n+1} = \frac{tz}{1 + (2 - t^2)z^2 + z^4}.$$

Now divide both sides by z, replace z^2 by z, and compare with the generating function of $tp_n(t^2)$. $\quad \square$

Theorem 3. *The nonzero spectrum of* OD_n *and of* AD_n *is the multiset*

$$\left\{ \pm 4 \cos \frac{k\pi}{2n+2} \cos \frac{l\pi}{2n+2} \; \middle| \; 1 \leq k \leq n, 1 \leq l \leq n \right\}.$$

The multiplicity of zero is $2n + 1$ *for* OD_n *and* $2n$ *for* AD_n.

Proof. The claim about the multiplicity of zero follows simply by counting the number of vertices of the Aztec diamonds and subtracting the $2n^2$ nonzero eigenvalues, so we need only consider the nonzero spectrum. By Theorem 1 it suffices to consider OD_n. Using the same kind of reasoning as in the second proof of Theorem 1, we see that $W[\mathrm{OD}_n, V_n^1]$ (where V_n^1 is the smaller part of the bipartition) is isomorphic to $P_n \times P_n$. (Figure 2 may be helpful again here.) By Lemma 2 it follows that the spectrum of $P_n \times P_n$ is

$$\left\{ 16 \cos^2 \frac{k\pi}{2n+2} \cos^2 \frac{l\pi}{2n+2} \; \middle| \; 1 \leq k \leq n, 1 \leq l \leq n \right\}.$$

The theorem now follows by Lemma 1. $\quad \square$

4. Complexity

The complexity of a periodic strip seems much less tractable than its spectrum. It is not well-behaved under taking walk graphs or graph products, so the techniques of our second proof of Theorem 1 do not seem to apply. The comparison of various algebraic invariants of OD_n and AD_n that are related to the complexity—e.g., the Tutte polynomial, the reliability polynomial, the spanning forests, or the characteristic polynomial of the Laplacian matrix—reveals no obvious generalization of Conjecture 2 to these contexts. Part of the reason for these difficulties seems to be the "global" nature of a spanning tree. As we shall see from our proof of Theorem 2, which we now present, the partial results we

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have exploit the fact that for periodic strips the enumeration of spanning trees can in a sense be "localized."

Proof of Theorem 2. We say that a subgraph F of G_n is a *grounded forest* of G_n if F is acyclic and if every vertex of G_n is connected to at least one vertex of S_{pn} by a path (possibly of zero length) in F. Given a grounded forest F, we can define an equivalence relation on the vertices of S_{pn} by writing $u \sim v$ if u and v are connected by a path in F; the resulting partition of S_{pn} is called the *type* of F. Note that spanning trees are just grounded forests whose type is the trivial partition.

If T is a spanning tree of G_{n+1} then we claim that the portion F_T of T that lies in G_n is a grounded forest of G_n. Clearly F_T is acyclic, so suppose that some vertex v of G_n is not connected to any vertex of S_{pn} by a path in F_T. Since the edges of G only connect vertices in *adjacent* blocks, it follows that there does not exist a path from u to any vertex of S_{pn} even if we are allowed to use *all* the edges of T—but this contradicts the fact that T spans G_{n+1}. Hence F_T is indeed a grounded forest as claimed.

Next let σ be any partition of $S_{(p+1)n}$. If F is a grounded forest of G_n, then we claim that the set of subsets of edges of

$$G(S_{pn} \cup S_{pn+1} \cup \cdots \cup S_{(p+1)n})$$

that extend F to a grounded forest of G_{n+1} of type σ depends only on the type of F, i.e., if we take a set of edges that extends F to a grounded forest of G_{n+1} of type σ, then replacing F by any other grounded forest of G_n of the same type will again produce a grounded forest of G_{n+1} of type σ. The proof of this is clear.

Let m be the number of partitions of S_0, and list the partitions of S_0 in some arbitrary order. This induces an ordering of the partitions of S_{pn} for all n by means of the periodic strip bijections. Let $\kappa_i(n)$ be the number of grounded forests of G_n whose type is the ith partition of S_{pn} under this ordering. We can express the conclusion of the previous paragraph by saying that there exist constants c_{ij} such that

$$\kappa_1(n+1) = c_{11}\kappa_1(n) + c_{12}\kappa_2(n) + \cdots + c_{1m}\kappa_m(n)$$
$$\kappa_2(n+1) = c_{21}\kappa_1(n) + c_{22}\kappa_2(n) + \cdots + c_{2m}\kappa_m(n)$$
$$\vdots$$
$$\kappa_m(n+1) = c_{m1}\kappa_1(n) + c_{m2}\kappa_2(n) + \cdots + c_{mm}\kappa_m(n)$$

for all $n > 0$. (The c_{ij} are independent of n because they depend only on

$$G(S_{pn} \cup S_{pn+1} \cup \cdots \cup S_{(p+1)n}),$$

and since G is a periodic strip these induced subgraphs are isomorphic for all n.) By multiplying everything by x^{n+1} and summing over all $n > 0$, we see that for every i the

generating function of $\kappa_i(n)$ is a rational function with denominator $\det(\delta_{ij} - xc_{ij})$. Thus the $\kappa_i(n)$ satisfy a linear recurrence with constant coefficients, as required.

To see that all conjugates satisfy the same linear recurrence, observe that the argument we gave to derive the matrix (c_{ij}) can be modified to yield a transition matrix M_1 (not necessarily square) between S_{pn} and S_{pn+1} (instead of between S_{pn} and $S_{(p+1)n}$). Similarly we can obtain transition matrices M_r between S_{pn+r-1} and S_{pn+r}. The matrix (c_{ij}) is then the product of p consecutive M's. Now observe that the matrix (c_{ij}) for a conjugate of G is just a product of the same p matrices but cyclically permuted. Such a cyclic permutation is just a similarity transformation, and thus it does not change the spectrum of (c_{ij}). Since the linear recurrence depends only on the coefficients of $\det(\delta_{ij} - xc_{ij})$, which in turn depends only on the spectrum of (c_{ij}), the theorem is proved. $\square$

5. Future work

The most obvious next task is to prove Conjecture 2. We remark that it might be possible to generalize Conjecture 2 to alternating strips. There are three essentially distinct alternating strips that are generated by $(3,3)$-bipartite graphs, and in each case the complexity ratio is $3/4$. It would be interesting to compute further examples to see if a simple ratio is always obtained and, if so, to formulate and prove an appropriate conjecture.

In some ways the most satisfactory resolution of Conjecture 2 would be a bijective proof. As we have already remarked, examination of connected subgraphs and of spanning forests of the Aztec diamonds did not reveal any simple pattern. Thus it seems that both acyclicity and connectedness are crucial features of Conjecture 2. Observe that OD_n and AD_n are "almost" planar duals of each other. It is well-known that planar duals have equal complexity, and moreover there is a simple bijection between the spanning trees: given a spanning tree, take all the edges in the dual that are not crossed. In this bijection, the roles of acyclicity and connectedness are interchanged as one passes to the dual. This suggests that these ideas might lead to the desired bijective proof of Conjecture 2.

Notice that the method used to prove Theorem 2 can be adapted to obtain recurrences for many other invariants of periodic strips, such as the reliability polynomial or the generating function for spanning forests. Perhaps this can be used to help resolve conjectures about these polynomials in the case of periodic strips.

Some of the ideas in this paper continue to be valid even if property 1 of periodic strips is weakened slightly to allow edges between vertices in the same block. This allows chessboards to be incorporated into our framework. Computation of the linear recurrence satisfied by the complexity sequence of chessboards has revealed the following striking observation.

Conjecture 3. *The coefficients of the linear recurrence satisfied by the complexity sequence of chessboard strips are palindromic.*

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In fact, it appears that palindromic coefficients are rather common. About half of the periodic strips of period one generated by $(3,3)$-bipartite graphs give rise to palindromic coefficients. In many of these cases the coefficients are also unimodal and alternate in sign. However, it is not clear from our data what the right generalization of Conjecture 3 is. For example, the linear recurrence of the Aztec diamonds does not have palindromic coefficients.

6. Acknowledgments

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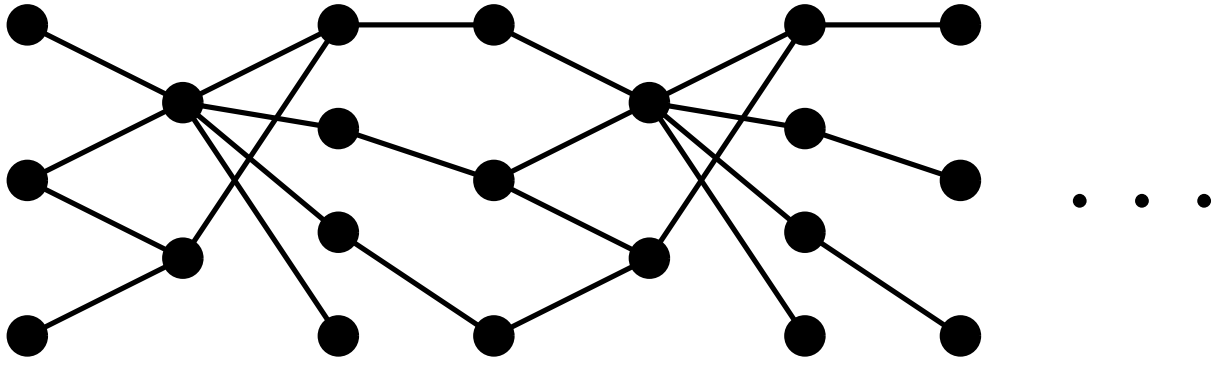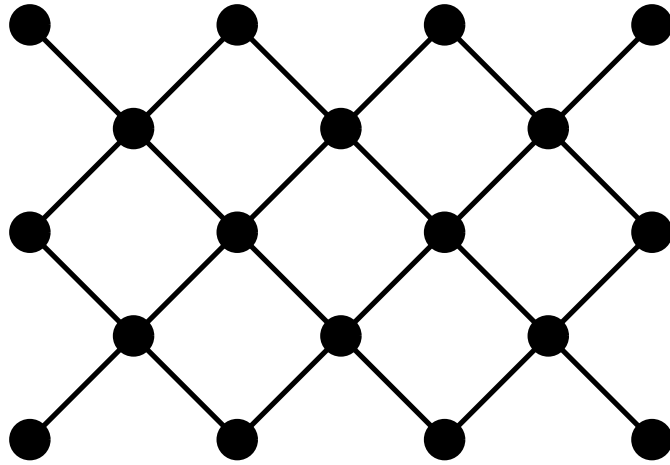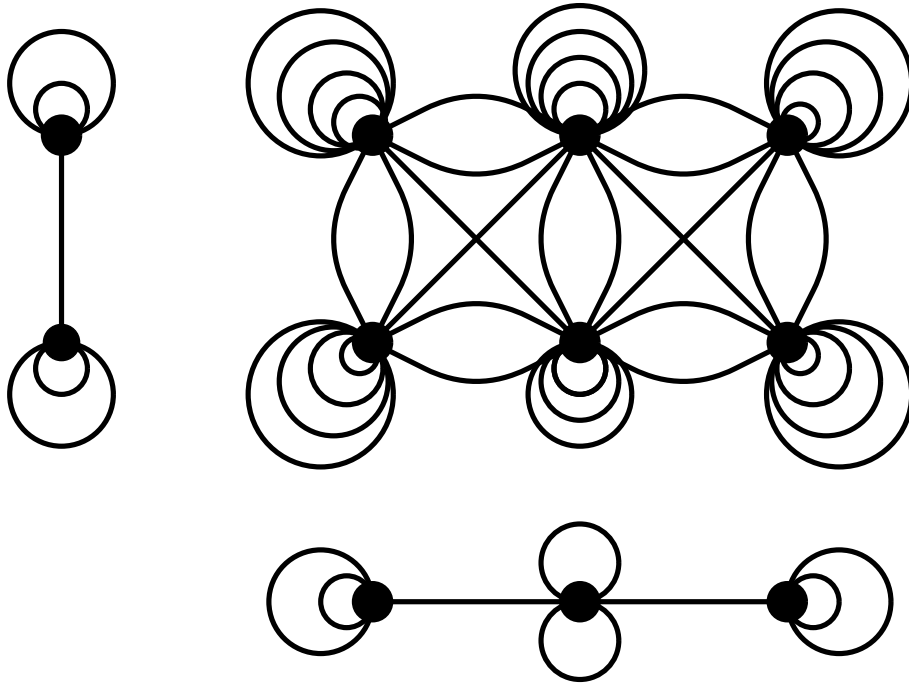
(a)
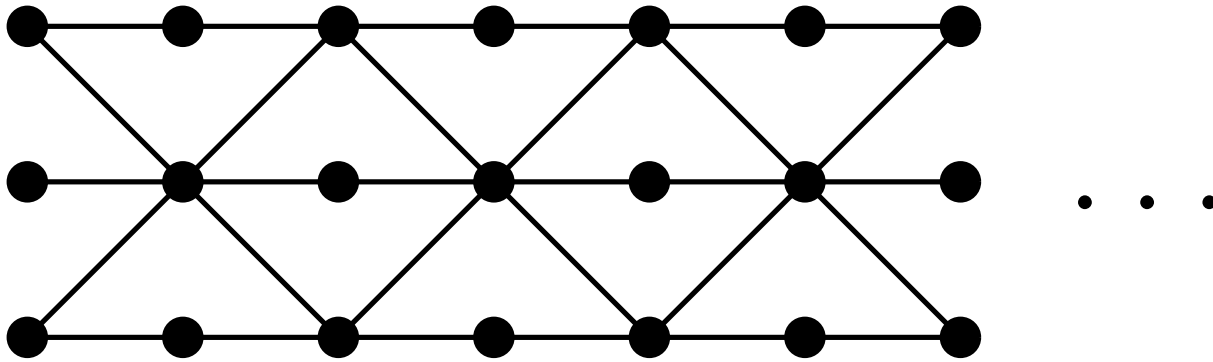
(b)

Figure 1. Examples of (a) a periodic strip and (b) an alternating strip.

(a)

(b)

Figure 2. Example of (a) part of an alternating strip and (b) the factorization of its walk graph.