This paper was written in the late 1990s and was even accepted for publication, but at the last minute, I discovered that the results had largely been superseded by Miklós Ruszinkó's paper, "On the upper bound of the size of the r-cover-free families," *J. Combin. Theory Ser. A* **66** (1994), 302–310. So we never published the paper, but I am making it available online in case anyone finds it interesting.

k-Sperner Families

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Abstract. Call a family $\mathcal{N}$ of subsets of a given finite set F *k-Sperner* if, given any $k+1$ distinct members $N_1, N_2, \ldots, N_{k+1}$ of $\mathcal{N}$, there exist $k+1$ elements $x_1, x_2, \ldots, x_{k+1}$ of F such that $x_i \in N_j$ if and only if $i = j$. The problem of maximizing the cardinality of k-Sperner families (or the equivalent dual problem of minimizing the cardinality of "k-separating" families) arises in electrical engineering, in the theory of all-optical networks. For $k = 1$ the problem is solved by the famous Sperner theorem, but for general k the problem is unsolved. We obtain upper and lower bounds.

1. Introduction

Call a family $\mathcal{N}$ of subsets of a finite set F *k-Sperner* if, given any $k+1$ distinct members $N_1, N_2, \ldots, N_{k+1}$ of $\mathcal{N}$, there exist $k+1$ elements $f_1, f_2, \ldots, f_{k+1}$ of F such that $f_i \in N_j$ if and only if $i = j$. The *k-Sperner problem* is the problem of finding the maximum cardinality of a k-Sperner family, given F and k.

We can obtain a "dual" formulation of the k-Sperner problem by interchanging the roles of sets and elements. Call a family $\mathcal{F}$ of subsets of a finite set N *k-separating* if, given any $k+1$ distinct elements $n_1, n_2, \ldots, n_{k+1}$ of N, there exist $k+1$ members $F_1, F_2, \ldots, F_{k+1}$ of $\mathcal{F}$ such that $n_i \in F_j$ if and only if $i = j$. It is easy to check that the k-Sperner problem is equivalent to the problem of finding the minimum cardinality of a k-separating family, given N and k.

A 1-Sperner family of subsets of F is just a family of subsets of F, none of which is contained in another. In other words, it is an antichain in the Boolean algebra of all subsets of F. The 1-Sperner problem is therefore completely solved by the famous Sperner theorem (see [6]): if $f = |F|$, then the desired maximum cardinality is

$$\binom{f}{\lfloor f/2 \rfloor}.$$

That the dual problem of 1-separating families is equivalent to Sperner's theorem was first observed by Spencer [8].

The k-Sperner problem is very natural, but it does not seem to have been studied before, even though entire books [1][2] have been written about generalizations of Sperner's theorem. (In [2] the term "k-Sperner" is used, but for a concept different from ours.) This paper is a first attempt at writing this missing chapter.

We were led to consider the k-Sperner problem as a result of a problem in electrical engineering: wavelength assignment in an all-optical network. The interested reader should see [5, Chapter 8] for more details. Here we will just say that F may be thought of as a set of frequencies and $\mathcal{N}$ as a set of transmitters. If the k-Sperner condition is satisfied, then any $k+1$ transmitters may operate simultaneously: for each transmitter there is a frequency on which it may be heard without noise from any of the other transmitters.

Unless otherwise stated, F will always denote a finite set, k a positive integer, $\mathcal{N}$ a k-Sperner family of F, f the cardinality of F, and n the cardinality of $\mathcal{N}$.

2. An Upper Bound

In this section we obtain an upper bound for n (given f and k).

DEFINITION. If $\mathcal{M}$ is any family of subsets of F, then the *k-wise union* of $\mathcal{M}$ is the multiset

$$\bigl\{M_1 \cup M_2 \cup \cdots \cup M_k \mid \{M_1, M_2, \ldots, M_k\} \text{ is a set of distinct members of } \mathcal{M}\bigr\}.$$

By "multiset" we mean that we keep track of multiplicity, so that the cardinality of the k-wise union is exactly $\binom{m}{k}$ (where $m = |\mathcal{M}|$). We say that the k-wise union is *multiplicity-free* if all $\binom{m}{k}$ members are actually distinct subsets of F.

THEOREM 1. *A family $\mathcal{M}$ of subsets of F is k-Sperner if and only if its k-wise union is multiplicity-free and 1-Sperner.*

Proof. Assume that $\mathcal{M}$ is k-Sperner. Let

$$X \stackrel{\text{def}}{=} M_1 \cup M_2 \cup \cdots \cup M_k \qquad \text{and} \qquad \tilde{X} \stackrel{\text{def}}{=} \tilde{M}_1 \cup \tilde{M}_2 \cup \cdots \cup \tilde{M}_k$$

be any two distinct members of the k-wise union of $\mathcal{M}$. (They might be equal as subsets of F since the k-wise union is a multiset.) Then at least one of the $\tilde{M}$'s is not equal to any of the M's. Without loss of generality, assume that $\tilde{M}_1$ has this property. Applying the k-Sperner property to the $k+1$ distinct sets

$$M_1, M_2, \ldots, M_k, \tilde{M}_1,$$

we conclude that there exists $\tilde{x} \in \tilde{M}_1 \subseteq \tilde{X}$ such that $\tilde{x} \notin X$. Similarly, there exists $x \in X$ such that $x \notin \tilde{X}$. Therefore, X and $\tilde{X}$ are distinct *as subsets of* F, and moreover neither is contained in the other. This proves one direction of the theorem.

Conversely, assume that the k-wise union of $\mathcal{M}$ is multiplicity-free and 1-Sperner. Given any $k+1$ members $M_1, M_2, \ldots, M_{k+1}$ of $\mathcal{M}$, let

$$X \stackrel{\text{def}}{=} M_1 \cup M_2 \cup \cdots \cup M_k \qquad \text{and} \qquad \tilde{X} \stackrel{\text{def}}{=} M_2 \cup M_3 \cup \cdots M_{k+1}.$$

By assumption, $X \not\subseteq \tilde{X}$, so there exists $x_1 \in M_1$ such that $x_1 \notin \tilde{X}$. By repeating this argument with suitable redefinitions of X and $\tilde{X}$, we can find, for each i, an x_i that is in M_i but not in any M_j with $j \neq i$. Hence $\mathcal{M}$ is k-Sperner. ∎

COROLLARY 1. *For any k-Sperner family $\mathcal{N}$ of subsets of F,*

$$\binom{n}{k} \le \binom{f}{\lfloor f/2 \rfloor}.$$

Proof. This follows immediately from Theorem 1 and Sperner's theorem. ∎

3. A Lower Bound

In this section we give a lower bound for the optimal value of n. The idea is not original with us; it is implicit in the electrical engineering literature, and it was independently communicated to the first author by Peter Bro Miltersen. We include the argument here because of its importance.

Theorem 2. *Assume that $f \geq 4$. Let q be the largest prime power such that $q^2 \leq f$. Then there exists a k-Sperner family of subsets of F of cardinality $q^{1+\lfloor (q-1)/k \rfloor}$.*

Proof. The assumption that $f \geq 4$ ensures that q exists. Let $\mathbf{F}_q$ be the finite field with q elements. Given a polynomial p with coefficients in $\mathbf{F}_q$, define

$$N(p) \stackrel{\text{def}}{=} \{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q \mid p(x) = y\}.$$

Thus $N(p)$ is the "graph" of p. Since $q^2 \leq f$, we may identify $\mathbf{F}_q \times \mathbf{F}_q$ with a subset of F, and so we may regard $N(p)$ as a subset of F.

Now let $d = \lfloor (q-1)/k \rfloor$ and define

$$\mathcal{N} \stackrel{\text{def}}{=} \{N(p) \mid p \text{ has degree at most } d\}.$$

We regard the zero polynomial as having degree zero. Two distinct polynomials of degree at most d over any field agree on at most d points of their domain, because their difference is a nonzero polynomial of degree at most d and hence has at most d zeroes. Since $q > d$, it follows that all the $N(p)$ listed in the definition of $\mathcal{N}$ are distinct, so the cardinality of $\mathcal{N}$ is q^{d+1}. Therefore, to prove the theorem, it suffices to show that $\mathcal{N}$ is k-Sperner.

The argument we just gave implies that the intersection of any two distinct members $N(p)$ and $N(\tilde{p})$ of $\mathcal{N}$ has cardinality at most d. Suppose we are given $k+1$ members $N(p_1), N(p_2), \ldots, N(p_{k+1})$ of $\mathcal{N}$. We claim that there exists $x_1 \in N(p_1)$ such that $x_1 \notin N(p_i)$ for any $i \neq 1$. For directly from its definition, $N(p_1)$ contains q elements, and it intersects each of the other $N(p_i)$ in at most d elements. But $q - kd \geq 1$ because $d = \lfloor (q-1)/k \rfloor$, so there must be some $x_1 \in N(p_1)$ that is not in any of the other $N(p_i)$. Similarly, for each j there exists $x_j \in N(p_j)$ such that $x_j \notin N(p_i)$ for any $i \neq j$. This shows that $\mathcal{N}$ is k-Sperner. ∎

Since squares of prime powers are rather sparsely distributed, f may be considerably larger than q^2, and then the construction of Theorem 2 is rather wasteful. This can be remedied somewhat by taking the elements of F in excess of $\mathbf{F}_q \times \mathbf{F}_q$ and finding a k-Sperner family of subsets of this excess. Combining this family with the k-Sperner family of subsets of $\mathbf{F}_q \times \mathbf{F}_q$ gives a larger family that is still k-Sperner. However, this procedure does not substantially improve the bound of Theorem 2.

4. A Limitation of the Method of Theorem 2

Let us shift to the dual perspective for a moment and think in terms of minimizing f given n and k. Using a few crude approximations, we find that Corollary 1 and Theorem 2 give lower and upper bounds (for the optimum value of f) of roughly $k \log(n/k)$ and $k^2 \log^2 n$ respectively. For a fixed value of k, the gap between these bounds is not all that large, but it turns out that for the application to electrical engineering that we have in mind, we need to consider the case when k is fairly large compared to n—say $k = n^{1/3}$ or even $k = n^{1/2}$. In this situation, the gap between our bounds is considerable.

It would be good news for our engineering application if $k \log(n/k)$ were closer to the truth than $k^2 \log^2 n$. Thus it is natural to try to sharpen the method used in proving Theorem 2. We will say a little more about this later, but here we wish to point out an inherent limitation.

THEOREM 3. *Let $\mathcal{N}$ be a family of subsets of F such that $|\mathcal{N}| \geq k+1$. Suppose there exists a positive integer d such that (a) the intersection of any two distinct members of $\mathcal{N}$ has at most d elements, and (b) every member of $\mathcal{N}$ has more than kd elements. Then $f \geq \binom{k+1}{2}$.*

Proof. Let $N_0, N_1, \ldots, N_k$ be distinct elements of $\mathcal{N}$. Let $m = kd$. Then $|N_0| \geq m$ by assumption (b). Also, by (a), N_1 intersects N_0 in at most d elements, so by (b), N_1 contains at least $m - d$ elements that are not contained in N_0. Continuing in this way, we see that each N_j contains at least $m - jd$ elements that are not contained in any of the preceding N_i. Therefore

$$\begin{aligned} f &\geq m + (m-d) + (m-2d) + \cdots + (m-kd) \\ &= (k+1)m - \frac{dk(k+1)}{2} \\ &= \frac{dk(k+1)}{2} \\ &\geq \binom{k+1}{2}. \quad \blacksquare \end{aligned}$$

Intuitively, Theorem 3 says that any construction that is based on the idea of controlling only *pairwise* overlaps cannot hope to reduce the exponent of k in the $k^2 \log^2 n$ bound. If we want to improve this bound substantially for the case when k is large compared to n, then we need a new idea.

5. 2-Sperner Families

It is easy to check that a family $\mathcal{M}$ of subsets of F is 2-Sperner if and only if no member of $\mathcal{M}$ is contained in the union of two other members of $\mathcal{M}$. In this form, the 2-Sperner condition has been defined before, but little seems to be known, e.g., Erdős and Kleitman [3, §5] say only that 2-Sperner families are exponentially small compared to 2^f (a fact that follows immediately from our Corollary 1).

In this section we give a bound for this special case that beats the general bound of Theorem 2. The idea is that since the k-Sperner condition has a lot of symmetry, choosing a random family of equal-sized subsets of F ought to be a good way of generating a k-Sperner family. In principle this approach should work for arbitrary k, but only for the case $k = 2$ have we been able to surmount the technical difficulties.

We outline the calculation before plunging into the details. Define a 2-*Sperner violation* of a family $\mathcal{M}$ of subsets of F to be a collection of three distinct members of $\mathcal{M}$ that fails to be 2-Sperner. First we estimate the probability $P(r)$ that a randomly chosen collection of three distinct subsets of size r fails to be 2-Sperner. By linearity of expectation, the expected number of 2-Sperner violations in a randomly chosen family of m subsets of size r is $\binom{m}{3}P(r)$. This means that there exists a family of m subsets of size r that has no more than $\binom{m}{3}P(r)$ 2-Sperner violations. Such a family may be made 2-Sperner by discarding one member of each 2-Sperner violation. Thus there exists a 2-Sperner family with $m - \binom{m}{3}P(r)$ members. Empirical testing or first-year calculus then tells us which value of m to choose to maximize the size of the 2-Sperner family.

Proposition 1. *The number of 3-tuples (M_1, M_2, M_3) of distinct r-element subsets of F such that $\{M_1, M_2, M_3\}$ is 2-Sperner is*

$$\binom{f}{r}\sum_i \binom{r}{i}\binom{f-r}{r-i}\left[\binom{f}{r} - 2\binom{f-r+i}{i} - \binom{2r-i}{r} + 2\binom{r}{i} + \binom{f-2r+2i}{2i-r} - \binom{i}{2i-r}\right].$$

Proof. (Sketch.) The subset M_1 may be chosen in $\binom{f}{r}$ ways. For each i from 0 to $r-1$, there are $\binom{r}{i}\binom{f-r}{r-i}$ ways to choose an M_2 that overlaps with M_1 in exactly i elements. Given a pair (M_1, M_2) that overlaps in exactly i elements, the number of ways to choose an M_3 so that $\{M_1, M_2, M_3\}$ is 2-Sperner is easily computed by an inclusion-exclusion argument to be the expression in brackets. Finally, the sum may be taken over all i, because the summand vanishes for i outside the range $0 \le i \le r-1$. ∎

Empirically, it seems that $r = \lfloor f/4 \rfloor$ yields large 2-Sperner families, so let us set $f = 4r$ from now on. If we divide the quantity in Proposition 1 by the total number of 3-tuples of distinct r-element subsets then we obtain

$$\frac{\sum_i \binom{r}{i}\binom{3r}{r-i}\left[\binom{4r}{r} - 2\binom{3r+i}{i} - \binom{2r-i}{r} + 2\binom{r}{i} + \binom{2r+2i}{2i-r} - \binom{i}{2i-r}\right]}{\left[\binom{4r}{r} - 1\right]\left[\binom{4r}{r} - 2\right]},$$

which is the probability that a randomly chosen set of three distinct r-element subsets is 2-Sperner. In the notation of our discussion preceding Proposition 1, this is just $1 - P(r)$.

These large expressions are cumbersome, so we seek an approximation.

LEMMA 1. *For $r > 2$, $P(r) \le r/(2.749)^r$.*

Proof. We are looking for an upper bound for $P(r)$, or equivalently a lower bound for $1 - P(r)$. Now

$$2\binom{r}{i} + \binom{2r+2i}{2i-r} - \binom{i}{2i-r} \ge 0,$$

so we may discard these terms. By Vandermonde summation,

$$\sum_i \binom{r}{i}\binom{3r}{r-i}\binom{4r}{r} = \binom{4r}{r}^2,$$

and

$$1 - \frac{\binom{4r}{r}^2}{\left[\binom{4r}{r}-1\right]\left[\binom{4r}{r}-2\right]} = \frac{2-3\binom{4r}{r}}{\left[\binom{4r}{r}-1\right]\left[\binom{4r}{r}-2\right]} \le 0,$$

so this may also be discarded, leaving us with

$$P(r) \le \frac{\left[2\sum_i \binom{r}{i}\binom{3r}{r-i}\binom{3r+i}{i}\right] + \left[\sum_i \binom{r}{i}\binom{3r}{r-i}\binom{2r-i}{r}\right]}{\left[\binom{4r}{r}-1\right]\left[\binom{4r}{r}-2\right]}.$$

Define

$$W_1(r) \stackrel{\text{def}}{=} \sum_i \binom{r}{i}\binom{3r}{r-i}\binom{3r+i}{i} \qquad \text{and} \qquad W_2(r) \stackrel{\text{def}}{=} \sum_i \binom{r}{i}\binom{3r}{r-i}\binom{2r-i}{r}.$$

Thanks to the Wilf-Zeilberger method [7], estimating these quantities is straightforward. The method yields the recurrence

$$c_2 W_1(r+2) = c_1 W_1(r+1) + c_0 W_1(r)$$

where

$$\begin{aligned}
c_0 &= 9(3922r^4 + 24804r^3 + 58711r^2 + 61650r + 24236)(3r+2)^2(3r+1)^2(r+1)^2,\\
c_1 &= 93390664r^{10} + 1057586168r^9 + 5283851606r^8 + 15310447400r^7\\
&\quad + 28439712648r^6 + 35318120852r^5 + 29640802742r^4 + 16573519332r^3\\
&\quad + 5902638604r^2 + 1208994384r + 108297072,\\
c_2 &= 9(3922r^4 + 9116r^3 + 7831r^2 + 2952r + 415)(3r+5)^2(3r+4)^2(r+2)^2.
\end{aligned}$$

Remarkably, $W_2(r)$ satisfies exactly the same recurrence. Now divide both sides of the recurrence by c_2. One checks by direct computation that

$$\frac{c_1}{c_2} = \frac{23812}{729} - R_1(r) \qquad \text{and} \qquad \frac{c_0}{c_2} = 1 - R_2(r),$$

where $R_1(r)$ and $R_2(r)$ are proper rational functions of r with nonnegative coefficients. This means that any function $W(r)$ that satisfies the recurrence

$$W(r+2) = \frac{23812}{729} W(r+1) + W(r) \tag{$*$}$$

and that equals or exceeds $2W_1(r) + W_2(r)$ at $r = r_0$ and $r = r_0 + 1$ will equal or exceed $2W_1(r) + W_2(r)$ for all $r \geq r_0$.

Let α be the larger root of the quadratic equation

$$x^2 - \frac{23812}{729} x - 1 = 0,$$

so that

$$W(r) \stackrel{\text{def}}{=} \frac{\alpha^r}{5}$$

satisfies the recurrence $(*)$. By direct computation one checks that $\alpha^r/5 \geq 2W_1(r) + W_2(r)$ for $r = 4$ and $r = 5$, so this inequality holds for all $r \geq 4$.

By Stirling's formula or a direct elementary argument one can show that

$$\frac{1}{5r}\left(\frac{4^4}{3^3}\right)^{2r} \leq \left[\binom{4r}{r} - 1\right]\left[\binom{4r}{r} - 2\right]$$

for $r \geq 4$. Combining this with the inequality of the previous paragraph we obtain

$$P(r) \leq \frac{\alpha^r/5}{\frac{1}{5r}\left(\frac{4^4}{3^3}\right)^{2r}} \leq \frac{r}{(2.749)^r}$$

for $r \geq 4$. Direct computation shows that the inequality of the theorem happens to hold for $r = 3$ as well. ∎

Theorem 4. *Assume that $f \geq 12$ and let $r = \lfloor f/4 \rfloor$. Then there exists a 2-Sperner family of subsets of F with cardinality*

$$\left\lceil \frac{2}{3} \left\lfloor \sqrt{\frac{2(2.749)^r}{r}} \right\rfloor \right\rceil.$$

Proof. We may assume that f is a multiple of 4, for if f is not a multiple of 4 then we can simply ignore $f - 4\lfloor f/4 \rfloor$ elements of F. Let

$$m = \left\lfloor \sqrt{\frac{2(2.749)^r}{r}} \right\rfloor.$$

If we randomly choose a family of m distinct r-element subsets of F, then the expected number of 2-Sperner violations is $\binom{m}{3}P(r)$, so there exists a family $\mathcal{M}$ with at most this many 2-Sperner violations. Pick such an $\mathcal{M}$ and let v be the number of 2-Sperner violations of $\mathcal{M}$. By Lemma 1,

$$\begin{aligned} v &\le \binom{m}{3} P(r) \le \binom{m}{3} \frac{r}{(2.749)^r} \le \frac{m^3}{6} \frac{r}{(2.749)^r} \\ &\le \frac{2(2.749)^r}{6r} \sqrt{\frac{2(2.749)^r}{r}} \cdot \frac{r}{(2.749)^r} = \frac{1}{3} \sqrt{\frac{2(2.749)^r}{r}}. \end{aligned}$$

Therefore

$$v \le \left\lfloor \frac{1}{3} \sqrt{\frac{2(2.749)^r}{r}} \right\rfloor \le \frac{1}{3} \left\lfloor \sqrt{\frac{2(2.749)^r}{r}} \right\rfloor = \frac{m}{3}.$$

We go through each 2-Sperner violation of $\mathcal{M}$ in turn, discarding one of the three subsets involved in the violation (if none of the three subsets has already been removed by previous discards). The remaining family $\mathcal{N}$ of subsets is 2-Sperner by construction, and it has at least $m - v \ge 2m/3$ members. ∎

From the dual perspective, Theorem 4 gives an upper bound of about $8 \log n$ for f, which improves the $O(\log^2 n)$ bound of Theorem 2 significantly and comes quite close to the lower bound of Corollary 1.

In principle there is no obstruction to applying our method with arbitrary k. The only problem is that we have not been able to prove that doing so actually improves the bound of Theorem 2. It is not difficult to generalize the exact formula of Proposition 1, but multiple summations arise that are not amenable to the Wilf-Zeilberger method.

6. Three Promising Constructions

Here we present three ideas for improving our lower bounds for n. As they stand, the ideas do not work, but we feel that they are promising nonetheless, and we hope that others will find ways to modify them so that they do work.

The first idea is that the construction in the proof of Theorem 2 makes use only of curves in the plane, so perhaps considering higher-dimensional varieties will yield better constructions. In higher dimensions the extra geometry may make it easier to say something about intersections of three or more varieties, thus circumventing the limitation described in Theorem 3.

We should mention, however, that merely moving to higher dimensions without any other new idea is unlikely to yield any improvement. Here is a heuristic calculation that illustrates the difficulty. Consider r-dimensional varieties in the projective space $\mathbf{P}^{2r}(\mathbf{F}_q)$. Think of each such variety as being defined by r homogeneous polynomials of homogeneous degree d and thus having degree d^r. By Bézout's theorem, we expect their pairwise intersections to have at most d^{2r} points. Each variety may be thought of as having about q^r points (e.g., Theorem 1 of [4]). If we follow the approach of the proof of Theorem 2 and choose our parameters so that

$$q^r - kd^{2r} \geq 1,$$

then this family of varieties will be k-Sperner. To estimate the number of varieties, note that the number of monomials of degree d in $2r+1$ variables is $\binom{d+2r}{2r}$. So the number of nonzero homogeneous polynomials is

$$q^{\binom{d+2r}{2r}} - 1$$

and the number of varieties is about

$$q^{r\binom{d+2r}{2r}}.$$

If we combine the equations $q^r = kd^{2r}$, $n = q^{rd^{2r}/(2r)!}$, and $f = q^{2r}$, then we obtain a rough estimate of

$$f \approx C(r)k^2 \log^2 n,$$

where $C(r)$ is some constant depending only on r. This is essentially the same bound as that of Theorem 2.

The second idea is that one can sometimes prove that distinct binary strings are indeed distinct by revealing rather small subsets of their bits. More precisely, we have the following proposition.

Proposition 2. *Given F and k as usual, let m be the largest integer such that*

$$2^k \binom{m}{k} \leq f.$$

Then there exists a k-Sperner family of subsets of F of cardinality 2^m.

Proof. For each positive integer d, let B_d be the set of all binary strings with exactly d digits. Let S be the set of all k-element subsets of $\{1, 2, \ldots, m\}$. By choice of m, we may identify $B_k \times S$ with a subset of F. For each element $b \in B_m$, let $N(b)$ be the set of all elements $(a, s) \in B_k \times S$ such that a is the substring of b consisting of the bits in the positions specified by S. For example, if $m = 3$, $k = 2$, and $b = 101$, then

$$N(b) = \big\{\big(10, \{1,2\}\big), \big(11, \{1,3\}\big), \big(01, \{2,3\}\big)\big\}.$$

The family $\mathcal{N}$ of all such $N(b)$ has cardinality 2^m, so it remains to show that $\mathcal{N}$ is k-Sperner. Given any $k+1$ members $N(b_1), N(b_2), \ldots, N(b_{k+1})$ of $\mathcal{N}$, we claim that there exists a k-element subset T of $\{1, 2, \ldots, m\}$ such that no two of the b_i's agree in *all* the bits in the positions specified by T. This may be proved by induction on k. The case $k = 1$ is clear. If $k > 1$, use induction to find $k-1$ bit positions that distinguish the first k binary strings $b_1, b_2, \ldots, b_k$. We are done if these positions are already enough to distinguish b_{k+1} from all the other b_i's. Otherwise, b_{k+1} agrees in all $k-1$ positions with at most one of the other b_i's—say b_1. Then we need to pick just one more bit position to distinguish b_{k+1} from b_1, completing the induction.

If we now let a_i be the substring of b_i specified by T, then $(a_i, T) \in N(b_i)$ for all i and $(a_i, T) \notin N(b_j)$ for all $j \neq i$ because of the choice of T. This shows that $\mathcal{N}$ is k-Sperner. ∎

The bound of Proposition 2 is fairly good for very small k, but it quickly becomes bad as k increases. However, there may be a way to improve the construction because usually one does not need anywhere near as many as k positions to distinguish between $k+1$ binary strings.

The third idea is to look for properties of sets that are not exactly the same as the k-Sperner condition but that are in some sense "close approximations." To illustrate the idea, call a family $\mathcal{M}$ of subsets of F *k-critical* if the union of any $k+1$ members of $\mathcal{M}$ equals all of F but the union of any k members of $\mathcal{M}$ is a proper subset of F. It is easy to see that k-critical families are k-Sperner, and k-critical families have the advantage of being much easier to handle than k-Sperner families, as the following proposition shows.

PROPOSITION 3. *Let $\mathcal{N}$ be a k-critical family of subsets of F. Then $f \geq \binom{n}{k}$, and this inequality is sharp.*

Proof. Consider the incidence matrix whose rows are indexed by the members of $\mathcal{N}$ and whose columns are indexed by the elements of F. That is, the (i, j) entry of this matrix equals one if the jth element of F lies in the ith member of $\mathcal{N}$, and equals zero otherwise. We claim that no column can contain more than k zeroes. For suppose there were; then there would exist $k+1$ rows that all had a zero in this column, and the union of the members of $\mathcal{N}$ corresponding to these rows would not be all of F, contradicting k-criticality.

Next we claim that given any set $R = \{R_1, R_2, \ldots, R_k\}$ of k rows, there must exist a column $C(R)$ with zeroes in all these rows. For otherwise, the union of the members of $\mathcal{N}$ corresponding to these k rows would be all of F, contradicting k-criticality. Since there are at most k zeroes in each column, $C(R)$ must have ones in every other row. This implies that if R and $\tilde{R}$ are two distinct k-element sets of rows, then $C(R)$ and $C(\tilde{R})$ must also be distinct. Hence there are at least $\binom{n}{k}$ columns.

This proves the inequality; to see that it is sharp, simply check that the incidence matrix with exactly the $\binom{n}{k}$ columns of the form demanded by the argument in the previous paragraph defines a k-critical family. ∎

Unfortunately, k-criticality is not a good enough approximation the the k-Sperner property to yield good bounds for the latter, but we are hopeful that some similar idea will work.

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