A Test-Waveform View of the Quadratic Design of FIR Filters

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Abstract

This paper suggests that when a test system in which an FIR filter is embedded is driven by a random-process input with a suitably chosen power spectrum, the system output power becomes a suitable quantity for constraint or minimization in a numerical optimization of the filter coefficients. Each such combination of test confuguration and test-input spectrum eventually corresponds to one constraint (or the objective function) in a quadratic optimization, which is here cast into the form of a semidefinite program for numerical optimization. Illustrated here by the design of a simple lowpass FIR filter, this method can be generalized to handle many types of filters and constraints.

Introduction

The design of an FIR filter to meet a given set of specifications has been approached in several ways [1–4] in the past. The goal is generally to find the filter coefficients that minimize the approximation error between the filter frequency response $H(f)$ and a desired response $D(f)$ subject to the given specifications. There are several ways to measure the approximation error. For example, the popular Chebyshev approximation [1] minimizes the maximum difference between $H(f)$ and $D(f)$, while the least-squares approach [2,4] minimizes instead the root-mean-square difference.

We present here a test-waveform approach to FIR filter design, a least-squares approach based on the optimization of a vector $\mathbf{h}$ of FIR-filter coefficients subject to suitably chosen power constraints. Translation of these power constraints into so-called *linear matrix inequalities* (LMIs) results in a *semidefinite program,* which can be effectively solved numerically. The selection of any particular power constraint reduces to an equivalent but more-intuitive selection of a conceptual test system and a corresponding test waveform. The test waveform is assumed to be a wide-sense-stationary random process, and the power constraint is an upper bound on the test-system output power in response to that test waveform.

The section below explains the test-waveform concept using a simple lowpass filter design as an example. The next section introduces semidefinite programs and LMIs, setting up the necessary background for the following discussion on translation of power constraints into LMIs. Finally, a numerical design example is presented to illustrate how the filter design can be represented and numerically solved as a semidefinite program.

The Test-Waveform Concept

To illustrate the test-waveform concept, consider the design of a simple lowpass FIR filter. Assume that we wish to design a real, FIR filter of order N with a frequency response approximating unity magnitude gain over a desired passband $[0, f_p]$ and an attenuated frequency-response magnitude over some stopband $[f_s, 0.5]$, where 0.5 is the Nyquist frequency. To design a filter with these features we first need to select suitable power constraints on the filter coefficient vector $\mathbf{h}$. This selection is simplified by using the test-waveform idea so that we instead choose two conceptual test-systems and their corresponding test-input waveforms, illustrated in fig. 1. The upper test setup in fig. 1 is chosen such that a constraint on its output power will induce the desired stopband, while a similar constraint on the output power in the lower test setup induces the desired passband characteristic. Note that in fig. 1 only the positive-frequency parts of the test-input power spectral densities $S_{x_1}(f)$ and $S_{x_2}(f)$ are shown, both of which are real, even functions of the normalized

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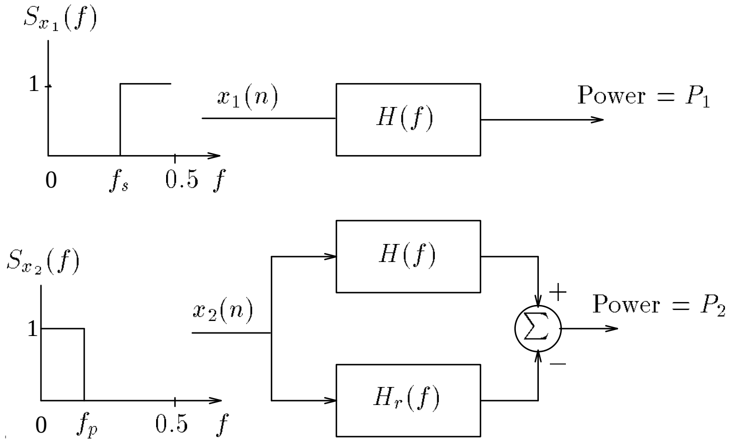

Figure 1: Stopband and passband optimization using the test-input method

frequency variable f.

The choice of the upper test setup in fig. 1 is explained first. The aim is to select a conceptual test system and a corresponding test-input waveform such that an upper bound on the system output power will induce an attenuated frequency-response magnitude in the stopband. A test system consisting simply of the FIR filter $H(f)$ being designed is chosen along with a wide-sense-stationary input waveform $x_1(n)$ that has high spectral density content in the stopband and zero spectral content elsewhere. Now, a low output power P_1 necessarily implies an attenuated magnitude response in the stopband. Therefore upper-bounding P_1 by some small number readily gives a power constraint that results in the desired stopband response.

The reasoning underlying the choice of the test setup in the bottom half of fig. 1 is similar. We selected the lower test system of fig. 1, which is like the upper test system of fig. 1 but modified to include an additional parallel connection containing a known reference FIR filter. The reference filter $H_r(f)$ is deliberately selected so that its frequency response has unit magnitude gain, the desired passband characteristic. Now, upperbounding the system output power P_2 by a small number will yield a magnitude response in the filter $H(f)$ that has approximately unit gain in $[0, f_p]$. This is because a small value for P_2 can only result when the responses of the two parallel filters are closely matched in the bands where $x_2(n)$ has high spectral density.

Here the test-waveform idea has simplified the selection of power constraints to a more-intuitive selection of test systems and corresponding test waveforms.

Semidefinite Programs

The lowpass-filter design problem can now be posed as a problem of optimizing the value of the vector $\mathbf{h}$ subject to the constraints above on powers P_1 and P_2. We choose to formulate this problem as a semidefinite program, which has the flexibility to represent problems involving both linear and positive-definite quadratic constraints, and which is effectively solved numerically. This section gives a brief introduction to semidefinite programs, following which we show how such a program can be used to formulate the lowpass-filter design problem. The variation of this idea to handle other filter types is straightforward.

Formally, a semidefinite program [5] specifies that a real n-vector $\mathbf{z}$ be found that minimizes a linear objective function $\mathbf{c}^T\mathbf{z}$ subject to constraints, each of which is a linear matrix inequality (LMI). An LMI [5, 6] is simply a requirement that some symmetric matrix function $\mathbf{M}(\mathbf{z}) = \mathbf{M}_0 + \mathbf{L}(\mathbf{z})$, where $\mathbf{M}_0$ is constant and $\mathbf{L}(\mathbf{z})$ is linear in $\mathbf{z}$, be positive semidefinite.[1] A semidefinite program therefore has the following form.

$$\min \mathbf{c}^T\mathbf{z}$$

subject to

$$\mathbf{M}_k(\mathbf{z}) \geq 0,\ k = 1, \ldots, N. \qquad (1)$$

where the inequality "$\geq$ 0" denotes positive semidefiniteness.

A semidefinite program is useful for FIR filter design because the underlying LMI form is flexible enough to represent commonly encountered

[1] Recall that matrix $\mathbf{A}$ is positive semidefinite if $\mathbf{y}^T\mathbf{A}\mathbf{y} \geq 0$ for every vector $\mathbf{y}$.

linear and positive-definite quadratic constraints accurately and elegantly. For example, the general LMI form $\mathbf{M}(\mathbf{z}) = \mathbf{M}_0 + \mathbf{L}(\mathbf{z}) \geq 0$ reduces to a linear constraint whenever $\mathbf{M}_0$ is a scalar constant and $\mathbf{L}(\mathbf{z})$ is scalar valued. Further, a constraint

$$\mathbf{z}^T\mathbf{D}\mathbf{z} \leq k$$

is a positive-definite quadratic constraint if $\mathbf{D} > 0$ and in that case is equivalent[2] to the LMI

$$\begin{bmatrix} k & \mathbf{z}^T \\ \mathbf{z} & \mathbf{D}^{-1} \end{bmatrix} \geq 0.$$

A semidefinite program can also be used to minimize a positive-definite quadratic objective subject to LMI constraints. The specification

$$\min\ \mathbf{z}^T\mathbf{B}\mathbf{z}$$

subject to

$$\begin{bmatrix} k & \mathbf{z}^T \\ \mathbf{z} & \mathbf{D}^{-1} \end{bmatrix} \geq 0$$

is converted to a standard semidefinite program by translating the quadratic objective to a second quadratic constraint using a new optimization variable t that is now minimized as the linear objective.

$$\min\ t$$

subject to

$$\begin{bmatrix} k & \mathbf{z}^T \\ \mathbf{z} & \mathbf{D}^{-1} \end{bmatrix} \geq 0$$
$$\begin{bmatrix} t & \mathbf{z}^T \\ \mathbf{z} & \mathbf{B}^{-1} \end{bmatrix} \geq 0$$

A semidefinite program is thus a very general form that can represent problems with linear and positive-definite quadratic constraints and objectives.

[2] The positive-semidefinite condition

$$\mathbf{M}(\mathbf{z}) = \begin{bmatrix} k & \mathbf{z}^T \\ \mathbf{z} & \mathbf{D}^{-1} \end{bmatrix} \geq 0$$

means that $\mathbf{x}^T\mathbf{M}(\mathbf{z})\mathbf{x} \geq 0$ for every $\mathbf{x} = \begin{pmatrix} m \\ \mathbf{n} \end{pmatrix}$, where $m, \mathbf{n}$ are a scalar and a vector respectively. Expand the left side of the inequality and complete the square to obtain the equivalent condition

$$(\mathbf{n} + m\mathbf{D}\mathbf{z})^T\mathbf{D}^{-1}(\mathbf{n} + m\mathbf{D}\mathbf{z}) + m^2(k - \mathbf{z}^T\mathbf{D}\mathbf{z}) \geq 0,$$

for every m and $\mathbf{n}$. When $m = 1$ and $\mathbf{n} = -m\mathbf{D}\mathbf{z}$, the inequality becomes just

$$k - \mathbf{z}^T\mathbf{D}\mathbf{z} \geq 0.$$

Thus a positive semidefinite $\mathbf{M}(\mathbf{z})$ implies $\mathbf{z}^T\mathbf{D}\mathbf{z} \leq k$. Conversely, $\mathbf{z}^T\mathbf{D}\mathbf{z} \leq k$ implies that $(\mathbf{n} + m\mathbf{D}\mathbf{z})^T\mathbf{D}^{-1}(\mathbf{n} + m\mathbf{D}\mathbf{z}) + m^2(k - \mathbf{z}^T\mathbf{D}\mathbf{z})$ is nonnegative if $\mathbf{D} > 0$. Therefore $\mathbf{z}^T\mathbf{D}\mathbf{z} \leq k$ and $\mathbf{D} > 0$ together imply that $\mathbf{M}(\mathbf{z})$ is positive semidefinite.

Power Constraints as LMIs

Formulating the lowpass-filter design presented earlier as a semidefinite program requires the translation of the associated power constraints into LMIs. It turns out, as shown below, that these power constraints are positive-definite quadratic constraints on the filter coefficients, and hence easily translated into LMIs. Below, a simple derivation for the filter output power is first presented, followed by LMI translation of a constraint that upperbounds this output power.

Let $y(n)$ be the output of an FIR filter in response to a real, wide-sense-stationary, random process $x(n)$. If the filter has N coefficients and impulse response $h(n)$, the output power becomes

$$\begin{aligned} E[|y(n)|^2] &= [h(n) \star R_x(n) \star h(-n)]_{n=0} \\ &= \sum_j \sum_m h(m)R_x(m-j)h(j) \\ &= \mathbf{h}^T\mathbf{R}\mathbf{h}, \end{aligned}$$

where $\star$ denotes convolution, where autocorrelation $R_x(n-k) = E[x(n)x(k)]$, and where $\mathbf{h} = [h(1), h(2), \ldots, h(N)]^T$ is the filter-coefficient vector. The kernel matrix $\mathbf{R}$ is a Töplitz matrix with element $\mathbf{R}_{mj} = R_x(m-j)$ and it is positive definite in practical cases. The inequality

$$\mathbf{h}^T\mathbf{R}\mathbf{h} \leq k$$

is thus a positive-definite quadratic constraint on $\mathbf{h}$ and easily translated into LMI

$$\begin{bmatrix} k & \mathbf{h}^T \\ \mathbf{h} & \mathbf{R}^{-1} \end{bmatrix} \geq 0.$$

Using the above idea, upper bounds on output powers P_1 and P_2 in fig 1 can be expressed as

$$\mathbf{h}^T\mathbf{R}_{x_1}\mathbf{h} \leq k_1 \tag{2}$$
$$(\mathbf{h} - \mathbf{h_r})^T\mathbf{R}_{x_2}(\mathbf{h} - \mathbf{h_r}) \leq k_2, \tag{3}$$

where $\mathbf{R}_{x_1}$ and $\mathbf{R}_{x_2}$ are the positive-definite autocorrelation matrices corresponding to inputs $x_1(n)$ and $x_2(n)$ respectively. Placing the nonzero coefficient in the center of reference-filter coefficient vector $\mathbf{h}_r = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ results, by symmetry, in a linear-phase design for the optimized filter.

A Simple Design Example

The following section presents the formulation and numeric optimization of a real, linear-phase, lowpass filter in which 21 coefficients are used to approximate a unity-gain passband $[0, 0.2]$ subject to a requirement that an input spectrally flat in stopband $[0.3, 0.5]$ be attenuated at least 70 dB.

To induce the stopband characteristic, use the upper test system of fig. 1 and concentrate the input spectral density in stopband $[0.3, 0.5]$. If the output power is limited to the input power less 70 dB, the power constraint becomes

$$\mathbf{h}^T \mathbf{R}_{x_1} \mathbf{h} \leq 10^{-\frac{70}{10}} R_{x_1}(0).$$

The unity-gain passband is induced by minimizing the output power of the lower test system with the input spectrum confined to passband $[0, 0.2]$.

The lowpass-filter design specification is then

$$\min\ (\mathbf{h} - \mathbf{h}_r)^T \mathbf{R}_{x_2} (\mathbf{h} - \mathbf{h}_r)$$

subject to

$$\mathbf{h}^T \mathbf{R}_{x_1} \mathbf{h} \leq 10^{-\frac{70}{10}} R_{x_1}(0),$$

which can be formulated as the semidefinite program

$$\min\ t$$

subject to

$$\begin{array}{rcl} \begin{bmatrix} 10^{-7} R_{x_1}(0) & \mathbf{h}^T \\ \mathbf{h} & \mathbf{R}_{x_1}^{-1} \end{bmatrix} & \geq & 0 \\ \begin{bmatrix} t & (\mathbf{h} - \mathbf{h}_r)^T \\ (\mathbf{h} - \mathbf{h}_r) & \mathbf{R}_{x_2}^{-1} \end{bmatrix} & \geq & 0. \end{array} \qquad (4)$$

However, attempted numeric solution of this semidefinite program fails due to ill-conditioning of the matrices $\mathbf{R}_{x_1}$ and $\mathbf{R}_{x_2}$, a consequence of the extremely high ratio of maximum to minimum eigenvalues in each matrix [7]. Recall that the elements of matrices $\mathbf{R}_{x_1}$ and $\mathbf{R}_{x_2}$ are samples of the autocorrelation sequences corresponding to inputs $x_1(n)$ and $x_2(n)$ respectively, so one way to reduce the eigenvalue ratio is to modify the power spectral densities of $x_1(n)$ and $x_2(n)$ by introducing a very small spectral floor in $S_{x_1}(f)$ and $S_{x_2}(f)$ across the bands that previously had zero power. This additional power in the test-input spectra may be viewed as out-of-band noise power and may have the undesirable side effect of degrading the desired frequency response. For this 21-coefficient lowpass filter, we removed the ill-conditioning by starting with an arbitrary high noise level and gradually decreasing it until frequency-response degradation was minimal. A noise floor 100 dB below the in-band level resulted.

This small change in the test-input spectra makes possible the numeric solution of the semidefinite program (4) using the LMITOOL-SP software package [8, 9]. Fig. (2) shows the resulting frequency-response magnitude.

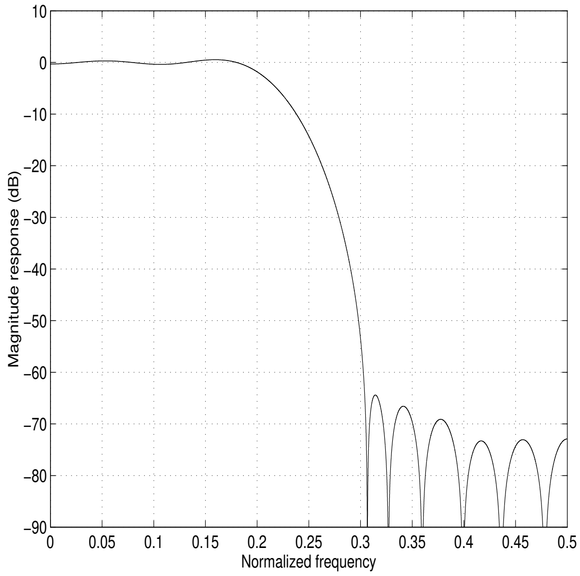

Figure 2: Magnitude response of a 21-coefficient lowpass FIR filter optimized to approximate unity gain in passband [0,0.2] subject to a 70 dB reduction in total power in stopband [0.3,0.5].

Though we deal explicitly above only with quadratic constraints, a semidefinite program also has the flexibility to incorporate constraints linear in the optimization variables. For example, suppose the passband in a real, N-tap linear-phase FIR filter is to approximate unity gain to within an error δ. Passband specification $|1 - H(f)| \leq \delta$ is treated as the two requirements

$$\begin{array}{rcl} 1 - H(f) & \leq & \delta \\ -(1 - H(f)) & \leq & \delta, \end{array}$$

each of which, at any fixed frequency f, is linear in the filter coefficients because $H(f) = h_0 + 2\sum_{n=1}^{N-1} h_n \cos 2\pi f n$. At each of a closely-spaced grid of frequencies $\{f_i\ :\ i = 1, \ldots, M\}$ then, $|1 - H(f)| \leq \delta$ can be represented as a pair of LMIs.

Summary

This paper presented a novel method of FIR filter design based on a test-waveform concept. The filter design problem is reduced to a simple, intuitive selection of appropriate test systems and test-input waveforms. The desired frequency-response features are induced by optimizing a vector of the filter coefficients subject to some suitable power constraints. Each of these power constraints is easily translated into an LMI, so that the filter design problem is readily formulated as a semidefinite program. The semidefinite-programming paradigm is highly flexible and can handle both linear and positive-definite quadratic constraints. Importantly, semidefinite programs are effectively solved numerically. A simple lowpass filter design was presented to illustrate this method.

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