ISCAS 2002: IEEE Int'l Symp. on Circuits and Systems
Scottsdale AZ, USA, May 26–29, 2002

COSET DECOMPOSITION IN LATTICES YIELDS SAMPLE-BLOCK NUMBER SYSTEMS

Jeffrey O. Coleman

Naval Research Laboratory
Washington DC
http://alum.mit.edu/www/jeffc

***Abstract*— The representation of a scalar by a sequence of digits weighted by powers of some radix is here generalized to vectors. The radix becomes a matrix, and its powers are applied to digits that are vectors taken from a lattice. Familiar notions of overflow and truncation error apply, and the familiar two's-complement approach to representing vectors on the "wrong side" of zero generalizes cleanly. Blocking 1D samples into vectors before conversion to these vector number systems can yield advantages in computational efficiency related to the packing efficiency of lattice points, much as data communication is improved by error-correcting block codes, a closely related topic.**

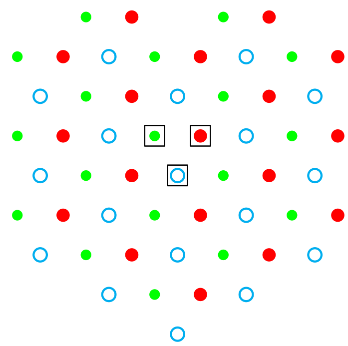


Figure 1: Multiplying the points of a 2D hexagonal lattice by a matrix R that rotates them by 30° and scales them by $\sqrt{3}$ yields a sublattice (circles) of one third the density. One set of coset representatives is $\left\{(0,0)^T, (1,\sqrt{3})^T, (-1,\sqrt{3})^T\right\}$ times half the point spacing of the original lattice.

1 INTRODUCTION

In the radix-r number system learned by every schoolchild, elements of a particular dense subset of the real numbers $\mathbb{R}$ are each expressed as a sum over a range $\{k\}$ of digit positions of terms from sets $\{r^k\mathcal{A}\}$, where—and here lies its beauty—$\mathcal{A}$ is a fixed alphabet of r digits chosen from the integers $\mathbb{Z}$. This paper generalizes: We express any N-vector from a particular set dense in $\mathbb{R}^N$ as a sum over digit positions $\{k\}$ of terms, each comprising a power R^k of radix matrix R times a digit from a fixed vector alphabet taken from a lattice $\Lambda \subset \mathbb{R}^N$. The concept will be close to obvious to anyone trained in abstract algebra, but this exposition aims to bring both the concept and practical aspects to DSP engineers.

Arithmetic algorithms analogous to the usual radix-r ones apply in these lattice-based number systems, but only rarely will real-time conversion between number systems at input and output make their use justifiable for signals in DSP systems. More often one might express only the coefficients of a large FIR filter or array combiner in a lattice-based number system in order to derive a hardware-efficient computational structure [1, 2, 3]. However, space limitations place such application matters beyond the scope of this paper, which therefore must focus on key basics of number systems in lattices.

This paper depends only on the most elementary concepts and notations of group theory, say from a standard senior-level abstract-algebra text like Herstein [4, pp. 26–54], and on the barest introduction to lattices, say pp. 4–5 of the bible of lattice theory, Conway and Sloane [5]. An alternative would be the introductory material in Forney's classic papers on the theory of coset codes [6, 7], to which the present topic is deeply connected. Finally, for readers neither easily frightened nor baffled, the following nanobackground might do.

This work was supported by the Office of Naval Research.

1.1 Background: Lattices, Sublattices, Cosets

Any subset of the real N-vectors $\mathbb{R}^N$ (including $\mathbb{R}^N$ itself) that is closed under addition and negation forms an an additive Abelian (commutative) group. If $\mathcal{N}$ and $\mathcal{G}$ are such groups with $\mathcal{N} \subset \mathcal{G}$, then $\mathcal{N}$ is a *normal subgroup* of $\mathcal{G}$. A *coset* of $\mathcal{N}$ in $\mathcal{G}$ is any translation $g + \mathcal{N} \triangleq \{g + n : n \in \mathcal{N}\}$ of a normal subgroup $\mathcal{N}$ by some $g \in \mathcal{G}$ called the *coset representative*. The *zero coset* is just $\mathcal{N}$. Example: The even integers $2\mathbb{Z} = \{2z : z \in \mathbb{Z}\}$ are a normal subgroup of the integers $\mathbb{Z}$, and the two distinct cosets of $2\mathbb{Z}$ in $\mathbb{Z}$ are the even integers $2\mathbb{Z}$ (the zero coset) and the odd integers $1 + 2\mathbb{Z}$.

A *coset decomposition* is a partition of a group into cosets of a normal subgroup, for example $\mathbb{Z} = (2\mathbb{Z}) \cup (1+2\mathbb{Z})$. The family of cosets of $\mathcal{N}$ in $\mathcal{G}$ is denoted by $\mathcal{G}/\mathcal{N}$, so $\mathbb{Z}/2\mathbb{Z} = \{2\mathbb{Z}, 1 + 2\mathbb{Z}\}$. The statement "even + odd = odd" is an example of *coset arithmetic*, addition and subtraction of cosets, and indeed, cosets $\mathcal{G}/\mathcal{N}$ always form an additive group. Notation $[\mathcal{G}/\mathcal{N}]$ denotes a set of *coset representatives*, one for each coset of $\mathcal{G}/\mathcal{N}$. We might, for example, set $[\mathbb{Z}/2\mathbb{Z}] = \{0, 1\}$. We generally express a coset decomposition like

$$\mathcal{G} = \bigcup_{g \in [\mathcal{G}/\mathcal{N}]} g + \mathcal{N}$$

concisely as $\mathcal{G} = [\mathcal{G}/\mathcal{N}] + \mathcal{N}$ to say that each member of group $\mathcal{G}$ is a sum of a coset representative from $[\mathcal{G}/\mathcal{N}]$ and an element of normal subgroup $\mathcal{N}$. Example: $\mathbb{Z} = [\mathbb{Z}/2\mathbb{Z}] + 2\mathbb{Z}$.

Any element of a coset can represent it, so coset representatives are not unique. Any odd integer represents them all: $1+2\mathbb{Z}=5+2\mathbb{Z}=-7+2\mathbb{Z}$. Generally $[\mathcal{G}/\mathcal{N}]$ denotes fixed but unspecified coset representatives, so $\mathbb{R}=[\mathbb{R}/\mathbb{Z}]+\mathbb{Z}$ holds whether $[\mathbb{R}/\mathbb{Z}]$ is interval $[0,1)$, interval $\left[-\frac{1}{2},\frac{1}{2}\right)$, etc.

A *lattice* is a set of form $M\mathbb{Z}^N$ of discrete points, so its elements are integer combinations of the columns of its *generating matrix* M. A *sublattice* is a subgroup of a lattice, so $2\mathbb{Z}^2$ is a sublattice of $\mathbb{Z}^2$. Coset decomposition is as before:

$$[\mathbb{Z}^2/2\mathbb{Z}^2] = \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\} \tag{1}$$

$$[\mathbb{R}^2/2\mathbb{Z}^2] = [0,2)\times[0,2).$$

Hexagonal lattice $\Lambda = M\mathbb{Z}^2$ of Fig. 1 is generated in 2D plane (x,y) or in 3D plane $x+y+z=0$ (different scales) with

$$M=\begin{pmatrix} 1 & -1 \\ \sqrt{3} & \sqrt{3} \end{pmatrix} \quad \text{or} \quad M=\begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{pmatrix}$$

respectively [5, pp. 5] and can be decomposed as shown into three cosets of sublattice $R\Lambda$ generated with, for the 3D case,

$$R=\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R^{-1}=\frac{1}{3}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

This R is singular, so this R^{-1} is its "inverse" only in that $R\,R^{-1}M=M$ and $R^{-1}RM=M$, which property will be useful later. Embedding this 2D lattice in a 3D space here conveniently eliminates irrational factors in these matrices.

2 LATTICE-BASED NUMBER SYSTEMS

To formally develop a number system based on an arbitrary lattice $\Lambda\subset\mathbb{R}^N$ and an arbitrary radix matrix R for which $R\Lambda$ is a sublattice of Λ, first use a coset decomposition to capture the idea of overflow past some desired leftmost digit K:

$$\mathbb{R}^N = \underbrace{[\mathbb{R}^N/R^{K+1}\Lambda]}_{\substack{\text{expressible in}\\ \text{digits } m\le K}} + \underbrace{R^{K+1}\Lambda}_{\substack{\text{overflow}\\ \text{error}}}. \tag{2}$$

With coset representatives fixed, this uniquely separates any vector in $\mathbb{R}^N$ into an overflow error in sublattice $R^{K+1}\Lambda$ and a remainder in $[\mathbb{R}^N/R^{K+1}\Lambda]$ expressible in digits through K.

After removing overflow error, we decompose the remainder into digits. Scale coset decomposition $\Lambda=[\Lambda/R\Lambda]+R\Lambda$ on the left by R^k to obtain $R^k\Lambda = R^k\,[\Lambda/R\Lambda]+R^{k+1}\Lambda$, and substitute this into coset decompositon $\mathbb{R}^N=[\mathbb{R}^N/R^k\Lambda]+R^k\Lambda$ to obtain $\mathbb{R}^N=[\mathbb{R}^N/R^k\Lambda]+R^k\,[\Lambda/R\Lambda]+R^{k+1}\Lambda$. When interpreted as a decomposition into cosets of $R^{k+1}\Lambda$ in $\mathbb{R}^N$, this yields

$$\underbrace{[\mathbb{R}^N/R^{k+1}\Lambda]}_{\substack{\text{expressible in}\\ \text{digits } m\le k}} = \underbrace{[\mathbb{R}^N/R^k\Lambda]}_{\substack{\text{expressible in}\\ \text{digits } m< k}} + R^k\,\underbrace{[\Lambda/R\Lambda]}_{\text{digit } k}. \tag{3}$$

Special case $k=K$ splits the remainder set $[\mathbb{R}^N/R^{K+1}\Lambda]$ in (2) into the possible contributions of digit K, the *most significant digit* (MSD), and a new remainder set $[\mathbb{R}^N/R^K\Lambda]$. Recursion on k in (3) then yields the contributions of digits $K-1, K-2, \dots, 1$. Reinterpreting (3) for $k=0$, the recursion stops with the *least significant digit* (LSD):

$$\underbrace{[\mathbb{R}^N/R\Lambda]}_{\substack{\text{expressible in}\\ \text{digits } m\le 0}} = \underbrace{[\mathbb{R}^N/\Lambda]}_{\substack{\text{truncation}\\ \text{error}}} + \underbrace{[\Lambda/R\Lambda]}_{\substack{\text{digit 0,}\\ \text{the LSD}}}. \tag{4}$$

Here we arbitrarily declare this remainder to be a truncation error, but we could have instead expressed it with digits $k=-1,-2,\dots$ by letting R^{-1} designate a matrix that inverts R in the linear span of Λ as in the hexagonal-lattice example at the end of the introduction. Stopping at $k=0$ avoids this subtlety, and since lattice Λ is arbitrary no generality is lost.

Assemble the complete representation from these pieces by beginning with overflow relationship (2) and substituting from recursion (3) for each of $k=K, K-1, \dots, 0$ to obtain

$$\mathbb{R}^N = \underbrace{[\mathbb{R}^N/\Lambda]}_{\substack{\text{truncation}\\ \text{error}}} + \underbrace{\left(\sum_{k=0}^{K} R^k\,[\Lambda/R\Lambda]\right)}_{\substack{\text{representation}\\ \text{in } K+1 \text{ digits}}} + \underbrace{R^{K+1}\Lambda}_{\substack{\text{overflow}\\ \text{error}}}. \tag{5}$$

To aid interpretation, first consider the $N{=}1$ dimensional example with lattice $\Lambda=\mathbb{Z}$, radix "matrix" set to scalar $R=2$, digit values $[\Lambda/R\Lambda]=[\mathbb{Z}/2\mathbb{Z}]$ fixed at $\{0,1\}$, and truncation errors $[\mathbb{R}^N/\Lambda]=[\mathbb{R}/\mathbb{Z}]=[0,1)$. Then (5) specializes to

$$\mathbb{R} = \underbrace{[0,1)}_{\substack{\text{truncation}\\ \text{error}}} + \underbrace{\left(\sum_{k=0}^{K} 2^k\,\{0,1\}\right)}_{\substack{K+1 \text{ digit}\\ \text{binary}}} + \underbrace{2^{K+1}\mathbb{Z}}_{\substack{\text{overflow}\\ \text{error}}}. \tag{6}$$

So any real number is expressible as a $K{+}1$ bit binary word just left of the binary point and truncation and overflow errors from not continuing further to the right and left respectively.

Remainder sets $[\mathbb{R}^N/R^k\Lambda]$ in (2), (3), and (4) are coarse, intermediate truncation errors. With $[\mathbb{R}^N/\Lambda]=[\mathbb{R}/\mathbb{Z}]$ set to $[0,1)$ in our 1D binary example, these remainder sets for $k=0,\dots,K+1$ are related by simple scale factors. But this relationship is both fragile, disappearing for example if we change $[\mathbb{R}/\mathbb{Z}]$ to $[-\frac{1}{2},\frac{1}{2})$ to make $[\mathbb{R}^N/\Lambda]=[\mathbb{R}/\mathbb{Z}]$ in (5) into a roundoff error, and prized because it enables a convenient iterative algorithm to perform the decomposition of (2), (3), (4), and hence (5). Return to the general setting to see how.

2.1 Left-to-Right Conversion

Suppose (3) relates consecutive members of sequence $[\mathbb{R}^N/\Lambda]$, $[\mathbb{R}^N/R\Lambda]$, $[\mathbb{R}^N/R^2\Lambda], \dots, [\mathbb{R}^N/R^{K+1}\Lambda]$. Then for $k=0,\dots,K$ any $\mathbf{x}_{k+1}\in[\mathbb{R}^N/R^{k+1}\Lambda]$ can be decomposed as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + R^k\lambda_k, \tag{7}$$

let $\mathbf{x}$ **be** the vector in $\mathbb{R}^N$ to be converted

$\mathbf{y} := \begin{bmatrix} \mathcal{M}_{lr}(\mathbf{x}) \\ \mathcal{Q}_{rl}(\mathbf{x}) \end{bmatrix}$

here $\mathbf{x} - \mathbf{y}$ **is** the $\begin{bmatrix} \text{overflow} \\ \text{truncation} \end{bmatrix}$ error

repeat $K+1$ **times:**

$\lambda := \begin{bmatrix} \mathcal{Q}_{lr}(\mathbf{y}) \\ \mathcal{M}_{rl}(\mathbf{y}) \end{bmatrix}$

here digits $\{\lambda\}$ appear in $\begin{bmatrix} \text{decreasing} \\ \text{increasing} \end{bmatrix}$ order of k

$\mathbf{y} := \begin{bmatrix} R\,(\mathbf{y} - R^K \lambda) \\ R^{-1}(\mathbf{y} - \lambda) \end{bmatrix}$

here $\begin{bmatrix} \mathbf{y} \\ R^{K+1}\,\mathbf{y} \end{bmatrix}$ **is** $\begin{bmatrix} R^{K+1} \times \text{truncation error} \\ \text{overflow error} \end{bmatrix}$

Figure 2: In this [left-to-right / right-to-left] conversion algorithm, successive intermediate vectors $\mathbf{y}$ computed correspond to $\mathbf{y}_k$ in the text beginning with $[\mathbf{y}_K/\mathbf{y}_0]$ and with k then [decreasing / increasing]. Which algorithm is preferred generally depends on the ease of implementing its digit production using [quantization in $\mathbb{R}^N$ / a modulo operation in lattice Λ].

where $\lambda_k \in [\Lambda/R\Lambda]$ is digit k and remainder $\mathbf{x}_k \in [\mathbb{R}^N/R^k\Lambda]$. To realize this decomposition step for $k=K$, define function $\mathcal{Q}_{lr} : [\mathbb{R}^N/R^{K+1}\Lambda] \rightarrow [\Lambda/R\Lambda]$ with $\mathcal{Q}_{lr}(\mathbf{x}_{K+1}) \triangleq \lambda_K$. Could this function realize the $k \neq K$ decompositions as well?

Recursion (3) creates nonunique set $[\mathbb{R}^N/R^{k+1}\Lambda]$ from a given $[\mathbb{R}^N/R^k\Lambda]$. Alternatively, left multiply coset decomposition $\mathbb{R}^N = [\mathbb{R}^N/R^k\Lambda] + R^k\Lambda$ by R to get $\mathbb{R}^N = R\,[\mathbb{R}^N/R^k\Lambda] + R^{k+1}\Lambda$ and its equivalent, the alternative recursion

$$[\mathbb{R}^N/R^{k+1}\Lambda] = R\,[\mathbb{R}^N/R^k\Lambda]. \tag{8}$$

Suppose this scaling holds for $k=0,\ldots,K$ as it did in special case (6) so that sets $\{R^{K-k}\,[\mathbb{R}^N/R^{k+m}\Lambda] : k=0,\ldots,K\}$ are identical and equal to $[\mathbb{R}^N/R^{K+m}\Lambda]$ for either $m \in \{0,1\}$. Left multiplying (7) by R^{K-k} then yields

$$\underbrace{R^{K-k}\mathbf{x}_{k+1}}_{\substack{(\cdot)\,\in\, R^{K-k}[\mathbb{R}^N/R^{k+1}\Lambda] \\ =\,[\mathbb{R}^N/R^{K+1}\Lambda]}} = \underbrace{R^{K-k}\mathbf{x}_k}_{\substack{(\cdot)\,\in\, R^{K-k}[\mathbb{R}^N/R^{k}\Lambda] \\ =\,[\mathbb{R}^N/R^{K}\Lambda]}} + R^K\lambda_k.$$

The decomposition on the right is now unique, and $\mathcal{Q}_{lr}(\,\cdot\,)$ is now applicable. Therefore, for any $k = 1,\ldots,K$,

$$\lambda_k = \mathcal{Q}_{lr}(R^{K-k}\mathbf{x}_{k+1})$$
$$R^{K-k}\mathbf{x}_k = R^{K-k}\mathbf{x}_{k+1} - R^K\lambda_k.$$

One more R yields $R^{K-(k-1)}\mathbf{x}_k = R\,(R^{K-k}\mathbf{x}_{k+1} - R^K\lambda_k)$, and $\mathbf{y}_k \triangleq R^{K-k}\mathbf{x}_{k+1}$ then leads to the algorithm of Fig. 2, which assumes a function $\mathcal{M}_{lr} : \mathbb{R}^N \rightarrow [\mathbb{R}^N/R^{K+1}\Lambda]$ that returns the remainder (modulo $R^{K+1}\Lambda$) in decomposition (2).

Consider the planar hexagonal lattice with three-row matrices M and R from the introduction and $\Lambda = M\mathbb{Z}^2 \subset \mathbb{R}^3$. Figure 3 plots lattice-plane components of $R^{K-k}\,[\mathbb{R}^N/R^{k+1}\Lambda]$ for $k=0,\ldots,K$ with $K=7$ and $[\mathbb{R}^N/\Lambda]$ set to $M\times$ the origin-centered unit square. Each diagram is built by $R^{K-k} \times$ recursion (3) from scaled, offset copies of the diagram above it. Scaling, implied by powers of R, rotates 30° clockwise and scales area by relative sublattice density $\frac{1}{3}$. Offsetting is by the scaled digits $R^K[\Lambda/R\Lambda]$ given by the columns of

$$(\,\text{●} \mid \text{●} \mid \text{●}\,) = R^K M \times \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \text{for } k < K, \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} & \text{for } k = K. \end{cases}$$

(Section 2.3 discusses this $k=K$ difference.) The view is from normal to the plane, so the axis spacing is 120°.

Apparently *the (normalized)* $[\mathbb{R}^N/R^k\Lambda]$ *tend to converge, no matter with what (reasonable)* $[\mathbb{R}^N/\Lambda]$ *recursion (3) is initialized* (not proved here). (*Aside:* Each $[\mathbb{R}^N/R^k\Lambda]$ is a pencil of translations of a line orthogonal to the 2D lattice, but the singular R zeroes the extraplanar component to leave $\{R^{K-k}[\mathbb{R}^N/R^k\Lambda]\}$ strictly in the lattice plane unless $k=K$.) The limiting $[\mathbb{R}^N/R^k\Lambda]$ scales per (8) and so can be used for left-to-right conversion if its decision boundaries are realizable, as when R is diagonal. For the case of decision boundaries that are hideous however, as in Fig. 3, we can derive another algorithm.

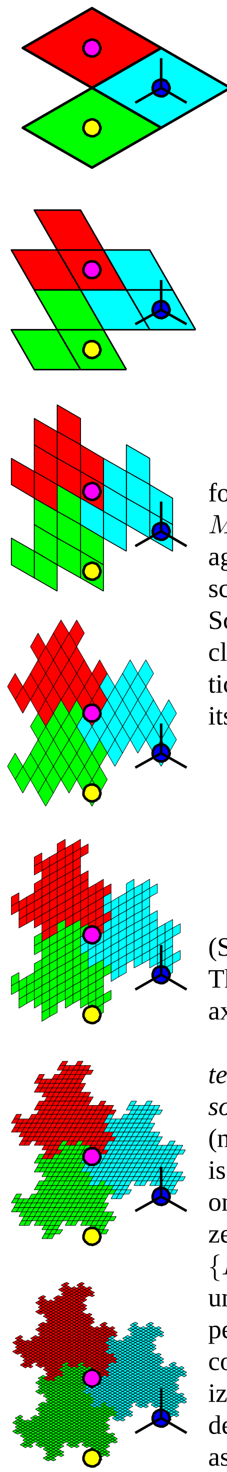
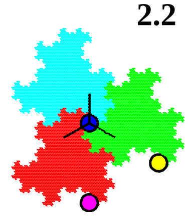


Figure 3: In "hex's complement" based on the Fig. 1 lattice/sublattice, digit k is determined according to the decision region into which a remainder in $[\mathbb{R}^N/R^{k+1}\Lambda]$ falls after higher-digit contributions are removed and it's scaled up by R^{K-k}. Decision diagrams at left for digit $k=0$ (top) through digit $k=K=7$ (bottom) were each determined by recursion (3) from the one just above using one digit set $[\Lambda/R\Lambda]$ for $k=0,\ldots,K-1$ and another for the $k=K$ "sign digit." This recursion from an $[\mathbb{R}^N/\Lambda]$ parallelogram approaches inconvenient limiting boundaries, so left-to-right conversion would be difficult.

2.2 Right-to-Left Conversion



If $R^{K-k}[\mathbb{R}^N/R^k\Lambda]$ depends on k, we cannot easily convert left to right. To convert right to left instead, use a different recursion, one that peels terms off of (5) in the other order:

$$\mathbb{R}^N = [\mathbb{R}^N/\Lambda] + \Lambda \tag{9}$$
$$R^k\Lambda = R^k\,[\Lambda/R\Lambda] + R^{k+1}\Lambda, \tag{10}$$

Again use two distinct functions to decompose $\mathbf{x} \in \mathbb{R}^N$. First use (9) to write $\mathbf{x} = \mathbf{e} + \mathbf{y}_0$, truncation error $\mathbf{e} \in [\mathbb{R}^N/\Lambda]$ and a remainder $\mathbf{y}_0 \in \Lambda$ expressible in digits $0, \ldots$ Then define quantization function $\mathcal{Q}_{rl}$: $\mathbb{R}^N \to \Lambda$ by $\mathcal{Q}_{rl}(\mathbf{x}) \triangleq \mathbf{y}_0$ and use (10) for $k = 0, \ldots, K$ to decompose each $R^k\,\mathbf{y}_k$ using a digit $\lambda_k \in [\Lambda/R\Lambda]$ and a remainder $\mathbf{y}_{k+1} \in \Lambda$:

$$R^k\,\mathbf{y}_k = R^k\lambda_k + R^k R\,\mathbf{y}_{k+1}. \tag{11}$$

Now define modulo-$R\Lambda$ function $\mathcal{M}_{rl}\colon \Lambda \to [\Lambda/R\Lambda]$ to realize the decomposition (11) for $k = 0$ with $\mathcal{M}_{rl}(\mathbf{y}_0) \triangleq \lambda_0$.

An inverse of R is needed for $k > 0$, but R can be singular, as in our example of a planar hexagonal lattice in 3D. If the linear spans of Λ and $R\Lambda$ are the same, however, there is a matrix, perhaps not unique, that we can call R^{-1} for convenience and for which $\lambda = R^{-1}R\,\lambda = R\,R^{-1}\lambda$ for any $\lambda \in \Lambda$, *i.e.* for $\lambda = Mz$ with any $z \in \mathbb{Z}$. (The R^{-1} at the end of the introduction for our hexagonal lattice met just this $R\,R^{-1}M = R^{-1}RM = M$ condition.) Each R in (11) operates in Λ, so left multiplying (11) by $R^{-k} = (R^{-1})^k$ yields

$$\underbrace{\mathbf{y}_k}_{(\cdot)\in\Lambda} = \underbrace{\lambda_k}_{(\cdot)\in[\Lambda/R\Lambda]} + \underbrace{R\,\mathbf{y}_{k+1}}_{(\cdot)\in R\Lambda}. \tag{12}$$

This a coset decomposition and therefore unique, so $\mathcal{M}_{rl}(\,\cdot\,)$ can again be used. Left multiply (12) by R^{-1} and rearrange to complete the right-to-left conversion algorithm of Fig. 2.

2.3 Generalizing Two's Complement

Until now, $[\Lambda/R\Lambda]$ has been assumed fixed, the same for all digits, but it need not be so. Consider the 1D binary example $\Lambda = \mathbb{Z}$ with $R = 2$ and $[\mathbb{R}^N/\Lambda] = [0, 1)$ again, and suppose that $[\Lambda/R\Lambda] = \{0, 1\}$ is used for digits $k = 0, \ldots, K-1$ as before but that $[\Lambda/R\Lambda] = \{0, -1\}$ is used now for digit K. Recursion (3) yields $[\mathbb{R}^N/R^K\Lambda] = [0, 2^K)$ as before, but then it goes on to yield $[\mathbb{R}^N/R^{K+1}\Lambda] = [\mathbb{R}^N/R^K\Lambda] + \{0, -1\} = [-2^K, 2^K)$ as the interval of reals representable in K bits (and a truncation error). This is the familar *two's complement* representation. When overflow is dropped or ignored in addition and subtraction of quantities represented as in (5), it is really cosets $[\mathbb{R}^N/R^{K+1}\Lambda]$ that are being added and subtracted, and it does not matter how the coset operands are represented. For multiplication, however, it is important to interpret the two's-complement sign bit as symbolically representing $\{0, -1\}$.

It works in just the same way in our example planar hexagonal lattice in 3D. In Fig. 3 the coset representatives $[\Lambda/R\Lambda]$ of recursion (3) were changed for the $k = K$ "sign digit" in order to place the zero-overflow region at the bottom of the figure symmetrically about zero, or almost so: The use of a zero-centered $[\mathbb{R}^N/\Lambda]$ at the top, more in the spirit of round-off than truncation, makes this symmetry only approximate. (How the choice of coset representatives should be changed for the sign digit for a general lattice is beyond the scope of this short paper.)

3 SUMMARY

This paper presented number systems that represent vectors in $\mathbb{R}^N$ using successive coset decompositions in a subgroup chain $\mathbb{R}^N/\Lambda/R\Lambda/R^2\Lambda/\ldots$, where $\Lambda = M\mathbb{Z}^N$ is a lattice and R is a radix matrix with $R\Lambda \subset \Lambda$. Truncating the infinite sequence of nested sublattices yields finite representations. A planar hexagonal lattice was featured for illustration, but all concepts, mathematics, and algorithms presented are general and apply regardless of dimension, lattice, or radix.

Two conversion algorithms were presented. Which is preferred generally depends on whether repeated decomposition into lattice and remainder components is simpler using quantization in $\mathbb{R}^N$ or using a modulo operation in the lattice, and that depends on the lattice and radix matrix. It can affect real-time performance, as lattice quantization is useful in DSP applications ranging from multidimensional $\Delta\Sigma$ modulation to high-performance computing structures for large FIR filters or the long correlators sometimes needed in radar pulse compression or spread-spectrum communications.

The focus in this paper was on the basics of overflow, truncation, and conversion from $\mathbb{R}^N$ because these are the first issues considered in any application. A subsequent paper will take up the performance questions of dynamic range and digit statistics and how they relate to the key question of choosing a combination of lattice and radix matrix.

REFERENCES

[1] J. O. Coleman, "Express Coefficients in 13-ary, Radix-4 CSD to Create Computationally Efficient Multiplierless FIR Filters," in *Proc. 2001 European Conf. on Circuit Theory and Design*.

[2] J. O. Coleman, "Cascaded Coefficient Number Systems Lead to FIR Filters of Striking Computational Efficiency," in *Proc. 2001 Int'l IEEE Conf. on Electronics, Circuits, and Systems*.

[3] J. O. Coleman, "Factored Computing Structures for Multiplierless FIR Filters Using Symbol-Sequence Number Systems in Linear Spaces," in *Proc. 2002 IEEE Int'l Conf. on Acoustics, Speech, and Signal Processing*.

[4] I. N. Herstein, *Topics in Algebra*, Wiley (NY), 2nd ed., 1975.

[5] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattice and Groups*, Springer-Verlag (New York), 2nd ed., 1988.

[6] G. D. Forney, Jr., "Coset codes—Part I: Introduction and geometrical classification," *IEEE Trans. Information Theory*, vol. 34, pp. 1123–1151, Sept. 1988.

[7] G. D. Forney, Jr., "Coset codes—Part II: Binary lattices and related codes," *IEEE Trans. Information Theory*, vol. 34, pp. 1152–1187, Sept. 1988.