The Spectral Products Created by Nonlinear Intersymbol Interference in NRZ Data

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Abstract — **In two-level noise-shaped D/A conversion, a delta-sigma modulator quantitizes a bandlimited discrete-time signal into a binary drive for a two-level D/A converter that then outputs a nominally clean NRZ data waveform with the quantization-noise spectrum shaped to be very low in the signal band. Nonlinear intersymbol interference in the final two-level conversion can, however, corrupt the signal band with intermodulation noise. Here a nonlinear model of that two-level conversion is used to derive its output spectrum for test data that are either periodic or random with i.i.d. bits, results that can enable laboratory measurement of model parameters and guide needed circuit improvements.**

I. Introduction

When BPSK pulse modulation or filtered two-level D/A conversion applies pulse shape $p(t)$ in a linear, shift-invariant (LSI) way to random data sequence $d(n)$ to create $y(t)=\sum_n d(n)\,p(t-nT)$, ordinary asynchronous spectral measurement estimates the continuous-time power spectral density of $y(t-t_0)$ given by

$$S_y(f) = S_d(fT)\,\frac{|P(f)|^2}{T} \tag{1}$$

(see Section A), where random delay t_0 is uniform on $[0,T)$ and independent of other variables. Here $S_d(\nu)$ is the power spectral density of the data, and $p(t) \leftrightarrow P(f)$ is a continous-time Fourier pair. In noise-shaped two-level D/A conversion, a $\Delta\Sigma$ modulator creates a ± 1 valued data sequence $d(n)=s(t)+n(t)$ with signal $s(t)$ and quantization noise $n(t)$ uncorrelated and corresponding to spectrally nonoverlapping $S_d(\nu)$ components so that $P(f)$ can eliminate the noise while preserving the signal. At high clock rates, however, circuit nonlinearities often invalidate this model and create signal-band intermodulation noise.

Intersymbol interference (ISI) without qualification usually means linear ISI that can be modeled in $p(t)\leftrightarrow P(f)$. *Nonlinear ISI* is data-related distortion that cannot be so modeled. (Linear and nonlinear ISI are collectively termed *deterministic jitter* by some component makers.) Here a variation of the nonlinear-ISI model in [1] is used to obtain three spectral results.

1. The model's output spectrum in terms of (a) the waveforms that parameterize the model and (b) easily estimated (in simulation) spectra and cross spectra of the data and various parity bits taken over the recent data history.
2. Spectra and cross spectra (b) when the $\Delta\Sigma$ modulator output is replaced with periodic test data. This enables lab measurement of spectral properties of waveforms (a).
3. Those key discrete-time spectra when the $\Delta\Sigma$ modulator output is replaced with a source of data that are random, independent, and equiprobably ± 1. In at least one key case, that of mismatched rise and fall time constants for a nominally NRZ signal, this allows parameterizing waveforms (a) to be derived exactly using a crosscorrelation. The insight provided by such an analytic solution can guide implementation improvement in the laboratory. (Due to space constraints the mismatched rise/fall derivation itself is omitted here. It will be presented elsewhere.)

II. The Nonlinear Conversion Model

This section recaps the nonlinear model of single-bit D/A conversion presented in [1]. Here it is both a little simpler and a little more general, and motivational material is largely omitted.

A. D/A Convolution

Define rate $1/T$ *D/A convolution* of a discrete-time signal $\mathbf{d}(n)$ and a continuous-time signal $\mathbf{c}(t)$ by

$$(\mathbf{d}\star\mathbf{c})(t) \triangleq T\sum_n \mathbf{d}(n)\,\mathbf{c}(t-nT)$$
$$(\mathbf{c}\star\mathbf{d})(t) \triangleq T\sum_n \mathbf{c}(t-nT)\,\mathbf{d}(n). \tag{2}$$

D/A convolution is not assumed commutative, as $\mathbf{d}(n)$ and $\mathbf{c}(t)$ may be dimensionally compatible matrices or vectors rather than scalars. The respective Fourier transforms are

$$T\sum_n \mathbf{d}(n)\,\mathbf{C}(f)\,\mathrm{e}^{-\mathrm{j}2\pi nfT} = T\mathbf{D}(fT)\,\mathbf{C}(f)$$
$$T\sum_n \mathbf{C}(f)\,\mathrm{e}^{-\mathrm{j}2\pi nfT}\,\mathbf{d}(n) = \mathbf{C}(f)\,T\mathbf{D}(fT).$$

B. A Nonlinear Model of Two-Level D/A Conversion

The system in Fig. 1 realizes a general time-invariant, causal map with finite nonlinear memory from a two-level discrete-time *signal input* $d(n)$ to a continuous-time analog output $y(t)$. A signal input $d(n)$, not shown, and the $\{0,1\}$-valued *control input* $x(n)$ on the lower right are affine (linear plus a constant) in each other. Signal input $d(n)$ is the value to be converted, and control input $x(n)$ is a $\{0,1\}$ digital encoding of that value. The encoding determines conversion type. A few possibilities:

conversion type	$d(n)\in$	*mapping*
bipolar	$\{\pm1\}$	$d(n)=2x(n)-1$
unipolar	$\{0,1\}$	$d(n)=x(n)$
bipolar, inverting	$\{\pm1\}$	$d(n)=1-2x(n)$
unipolar (noninv), scaled by two	$\{0,2\}$	$d(n)=2x(n)$

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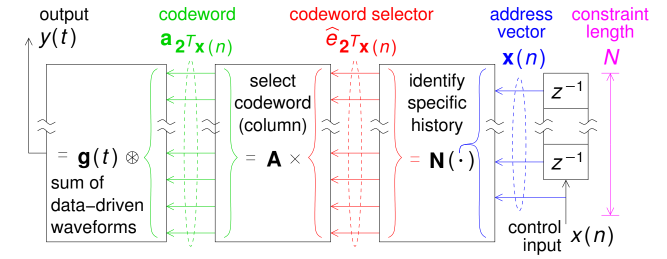


Fig. 1. A nonlinear model [1] of data conversion. The delay line creates address vector $\mathbf{x}(n)$ from binary control input $x(n)$. Nonlinearity $\mathbf{N}(\cdot)$ outputs standard unit vector $\widehat{e}_i$ (numbered from zero) whenever $\mathbf{2}^T\mathbf{x}(n) = i$, where $\mathbf{2}^T = (2^{N-1}, \ldots, 2^1, 2^0)$, and $\widehat{e}_i$ selects column i of matrix $\mathbf{A}$ to dot with row vector $\mathbf{g}(t)$ to create a single term of output $y(t) = T\sum_n \mathbf{g}(t-nT)\,\mathbf{AN}(\mathbf{x}(n))$.

The first of these was realized by the model in [1] (with $\mathbf{A}$ transposed relative to $\mathbf{A}$ here). The last two of these are used below to illustrate the slightly greater generality of the model here.

The system in Fig. 1 is laid out right to left to correspond to the usual order in which consecutive matrix operations on column vectors are written. Operation is straightforward. From control input $x(n)$, a *delay line* creates an *address vector*

$$\mathbf{x}(n) = \begin{bmatrix} x(n-(N-1)) \\ \vdots \\ x(n-1) \\ x(n) \end{bmatrix},$$

of length N, the model's *constraint length*, the maximum extent of its nonlinear memory. (Delay-line reversal relative to [1] simplifies construction below.) The 2^N possible addresses $\mathbf{x}(n)$ reflect 2^N distinct recent input histories and are mapped by nonlinear map $\mathbf{N}\colon \{0,1\}^N \to \mathbb{R}^{2^N}$, where $\mathbf{N}(\mathbf{x}) = \widehat{e}_{\mathbf{2}^T\mathbf{x}}$ and

$$\mathbf{2}^T = \begin{bmatrix} 2^{N-1} & \cdots & 2^2 & 2 & 1 \end{bmatrix}^T,$$

to the 2^N standard unit vectors $\widehat{e}_0, \ldots, \widehat{e}_{2^N-1}$ to create *codeword selector* $\widehat{e}_i = \mathbf{N}(\mathbf{x}(n))$. The next step selects *codeword* $\mathbf{a}_i = \mathbf{A}\widehat{e}_i$. The elements of $\widehat{e}_i$ are zero except for a single 1 in position i, so $\mathbf{a}_i$ is just column i of *code matrix* $\mathbf{A} = \begin{bmatrix}\mathbf{a}_0, \ldots, \mathbf{a}_{2^N-1}\end{bmatrix}$. System output $y(t)$, at the upper left in Fig. 1, is given by rate $1/T$ D/A convolution of the sequence of selected codewords with *waveform row vector* $\mathbf{g}(t)$. Informally we can write $y(t) = \mathbf{g}(t) \star \mathbf{AN}(\mathbf{x}(n))$ and understand it to mean the more formal

$$\mathbf{a}(n) = \mathbf{AN}(\mathbf{x}(n)) \tag{3}$$
$$y(t) = (\mathbf{g} \star \mathbf{a})(t), \tag{4}$$

where at each n the value of *codeword sequence* $\mathbf{a}(n)$ is some codeword $\mathbf{a}_i$. Alternatively, we can expand (3) and (4) and write

$$y(t) = T\sum_n \mathbf{g}(t-nT)\,\mathbf{AN}(\mathbf{x}(n)).$$

Given a constraint length N, little is variable in the construction of code matrix $\mathbf{A}$ (below). The model is tailored primarily by choice of waveform row vector $\mathbf{g}(t)$.

Code matrix $\mathbf{A}$ is constructed to set the first two elements of vector $\mathbf{a}(n)$ to unity and signal input $d(n)$ respectively so that in terms of $g_{\text{clock}}(t)$ and $g_{\text{LTI}}(t)$, the first two elements of $\mathbf{g}(t)$,

TABLE 1

CODE MATRIX $\mathbf{A} = \mathbf{CXH}$ FOR $N = 3$ AND TWO CHOICES OF $\mathbf{C}$.

key	
·	0
+	1
+	2

$\mathbf{XL}$			$\mathbf{L}^T$								$\mathbf{L}^T$							
			·	·	·	·	+	+	+	+	·	·	·	·	+	+	+	+
			·	·	+	+	·	·	+	+	·	·	+	+	·	·	+	+
			·	+	·	+	·	+	·	+	·	+	·	+	·	+	·	+
·	·	·	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
·	·	+	+	−	+	−	+	−	+	−	·	**+**	·	**+**	·	**+**	·	**+**
·	+	+	·	**+**	**+**	·	·	**+**	**+**	·	·	**+**	**+**	·	·	**+**	**+**	·
+	·	+	·	**+**	·	**+**	**+**	·	**+**	·	·	**+**	·	**+**	**+**	·	**+**	·
+	+	+	·	**+**	**+**	·	**+**	·	·	**+**	·	**+**	**+**	·	**+**	·	·	**+**
			$\mathbf{A} = \mathbf{C}_{\{\pm1\}}\mathbf{XH}$								$\mathbf{A} = \mathbf{C}_{\{0,2\}}\mathbf{XH}$							

$$\mathbf{g}(t) = \begin{bmatrix} g_{\text{clock}}(t) & g_{\text{LTI}}(t) & g_{011}(t) & g_{101}(t) & g_{111}(t) \end{bmatrix}$$

$$\begin{aligned} y(t) = \quad & T\sum_n g_{\text{clock}}(t-nT) \\ & + T\sum_n d(n)\, g_{\text{LTI}}(t-nT) \\ & + \begin{array}{l}\text{terms nonlinear in } d(n) \text{ and} \\ \text{involving other elements of } \mathbf{g}(t).\end{array} \end{aligned} \tag{5}$$

The first term comprises spectral lines at harmonics of *clock rate* $1/T$. The second is LSI in signal input $d(n)$, so the $g_{\text{LTI}}(t)$ component of waveform row vector $\mathbf{g}(t)$ models ideal D/A conversion. All other components model conversion nonlinearities.

C. Constructing the Code

Elements of vectors and matrices are numbered from zero. Construction begins with a $2^N \times N$ label matrix $\mathbf{L}$ that enumerates $0, 1, 2, \ldots, 2^N-1$ in binary. For example, when $N=3$ and denoting 0 and 1 by $\cdot$ and $+$ respectively,

$$\mathbf{L} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & + \\ \cdot & + & \cdot \\ \cdot & + & + \\ + & \cdot & \cdot \\ + & \cdot & + \\ + & + & \cdot \\ + & + & + \end{bmatrix}.$$

Defining map ϕ to return -1 or 1 according as its scalar argument is odd or even respectively and to operate elementwise on vectors and matrices, $\mathbf{H} \triangleq \phi(\mathbf{LL}^T)$ is a $2^N \times 2^N$ Hadamard matrix. Of even-numbered rows of $\mathbf{H}$ only row zero will be needed here [1], so construct row-excision matrix $\mathbf{X}$ to transfer the desired rows to $\mathbf{XH}$. When $N=3$ for example,

$$\mathbf{X} = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \end{bmatrix}.$$

Finally, let the code matrix be given by

$$\mathbf{A} \triangleq \mathbf{CXH}. \tag{6}$$

Row one of $2^{N-1}+1$ square *row-combining matrix* $\mathbf{C}$ determines row one of $\mathbf{A}$ and hence the conversion type, and other rows of $\mathbf{C}$ simply give elements of waveform row vector $\mathbf{g}(t)$ natural interpretations. Table 1 gives $\mathbf{A}$ for $N=3$ and two choices of $\mathbf{C}$. Denoting -1, 0, 1 by $-$, $\cdot$, and $+$ respectively,

$$\mathbf{C}_{\{\pm1\}} = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot \\ + & \cdot & - & \cdot & \cdot \\ + & \cdot & \cdot & - & \cdot \\ + & \cdot & \cdot & \cdot & - \end{bmatrix} \overset{\longleftarrow\text{row one}\longrightarrow}{;} \quad \mathbf{C}_{\{0,2\}} = \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot \\ + & - & \cdot & \cdot & \cdot \\ + & \cdot & - & \cdot & \cdot \\ + & \cdot & \cdot & - & \cdot \\ + & \cdot & \cdot & \cdot & - \end{bmatrix}.$$

If $\mathbf{C} = \mathbf{C}_{\{\pm 1\}}$, row one (row $\cdot\cdot+$) of code matrix $\mathbf{A}$ yields bipolar, inverting conversion. Changing to $\mathbf{C} = \mathbf{C}_{\{0,2\}}$ adds a DC output bias and inverts the $g_{\mathrm{LTI}}(t)$ drive, yielding unipolar (noninverting) conversion scaled by two. In either case, in Table 1 columns of $\mathbf{A}$ are codewords and rows of $\mathbf{A}$ except the first make corresponding codeword elements into parities over selected address bits. Column labels from $\mathbf{L}^T$ enumerate possible address vectors, and row labels from $\mathbf{XL}$ specify address bits for the implied parity computations. Rows zero and one of $\mathbf{A}$ drive clock-line and LTI waveforms in $\mathbf{g}(t)$ respectively, as in (5). Other rows add nonlinear components to the output.

III. Output Power Spectra for Specific Inputs

The system output spectrum is derived for periodic and random control inputs $x(n)$ after the background presented next,

A. The Second-Order Statistics of D/A Convolution

This section presents, as background, a slight generalization of an old result [2] that is widely applicable in D/A conversion and data transmission but that is seldom presented cleanly. The variables are local to this self-contained development.

Define continuous-time vector $\mathbf{y}_i(t) = (\mathbf{h}_i \star \mathbf{x}_i)(t - t_0)$ for each of $i = 1, 2$ in terms of discrete-time random vector $\mathbf{x}_i(n)$ and LTI system response $\mathbf{h}_i(t)$, which is in general a matrix. Let unknown system clock phase t_0 be independent of other random quantities and uniformly distributed on some semi-open interval $\mathcal{I}$ of width T. Using a conditional expectation, the matrix-valued crosscorrelation function for $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ is given by

$$\begin{aligned} &R_{\mathbf{y}_1\mathbf{y}_2}(\tau) \triangleq E\big(E\big(\mathbf{y}_1(t)\,\mathbf{y}_2^H(t-\tau)\,\big|\,t_0\,\big)\big) \\ &= \frac{1}{T}\int_{\mathcal{I}} T^2 \sum_{n,m} \begin{array}{l} \mathbf{h}_1(t - t_0 - nT) \\ E\big(\mathbf{x}_1(n)\,\mathbf{x}_2^H(m)\,\big) \\ \mathbf{h}_2^H(t - \tau - t_0 - mT)\,dt_0 \end{array} \end{aligned}$$

using $(\,\cdot\,)^H$ for Hermitian transpose. Suppose matrix-valued crosscorrelation $E\big(\mathbf{x}_1(n)\,\mathbf{x}_2^H(m)\,\big)$ is stationary and written as $R_{\mathbf{x}_1\mathbf{x}_2}(n-m)$. Defining $k = n-m$ and $s = t - t_0 - mT$ yields

$$R_{\mathbf{y}_1\mathbf{y}_2}(\tau) = \sum_m \int_{t-mT-\mathcal{I}} T \sum_k \mathbf{h}_1(s - kT)\,R_{\mathbf{x}_1\mathbf{x}_2}(k)\,\mathbf{h}_2^H(s - \tau)\,ds.$$

The regions of integration for different m partition the reals, so

$$\begin{aligned} R_{\mathbf{y}_1\mathbf{y}_2}(\tau) &= \int T \sum_k \mathbf{h}_1(s - kT)\,R_{\mathbf{x}_1\mathbf{x}_2}(k)\,\mathbf{h}_2^H(s - \tau)\,ds \\ &= \int (\mathbf{h}_1 \star R_{\mathbf{x}_1\mathbf{x}_2})(s)\,\mathbf{h}_2^H(s - \tau)\,ds \\ &= (\mathbf{h}_1 \star R_{\mathbf{x}_1\mathbf{x}_2} \star \mathbf{h}_2')(\tau), \end{aligned} \tag{7}$$

where $\mathbf{h}_2'(t) \triangleq \mathbf{h}_2^H(-t)$. The rate $1/T$ D/A convolution and the continuous-time convolution are associative. Either can be first.

The cross spectrum is the crosscorrelation's Fourier transform $\mathrm{S}_{\mathbf{y}_1\mathbf{y}_2}(f) = \mathbf{H}_1(f)\,T\mathrm{S}_{\mathbf{x}_1\mathbf{x}_2}(fT)\,\mathbf{H}_2^H(f)$. Or, in the special case with $\mathbf{x}_i(n) = \mathbf{x}(n)$, $\mathbf{h}_i(t) = \mathbf{h}(t)$, and $\mathbf{y}_i(t) = \mathbf{y}(t)$,

$$\mathrm{S}_{\mathbf{y}}(f) = \mathbf{H}(f)\,T\mathrm{S}_{\mathbf{x}}(fT)\,\mathbf{H}^H(f). \tag{8}$$

Fig. 2. If control input $x(n)$ has period M then address vector $\mathbf{x}(n)$ and codeword sequence $\mathbf{a}(n) = \mathbf{AN}(\mathbf{x}(n))$ do as well. The same sequence $\mathbf{a}(n)$ results if $\mathbf{a}(n) = \mathbf{A}_p\,\widehat{e}_{i(n)}$ with $i(n)$ cycling through $M-1\ldots 0$ with a random starting phase and with matrix $\mathbf{A}_p$ (right) constructed from columns of $\mathbf{A}$ (here $\mathbf{C}_{\{\pm1\}}\mathbf{XH}$) corresponding right to left to one period of $\mathbf{x}(n)$.

an $\mathbf{x}(n)$	+	·	·	+	·	+	+	·	+	·
period	·	+	·	·	+	·	+	+	·	+
XL	+	·	+	·	·	+	·	+	+	·
· · ·	+	+	+	+	+	+	+	+	+	+
· · +	−	+	−	+	+	+	+	−	−	+
· + +	+	+	+	·	+	+	+	·	+	+
+ · +	·	·	+	+	·	·	+	+	·	·
+ + +	·	+	+	+	+	·	·	·	·	+

A related result on D/A-conversion output-input crosscorrelation is easily derived directly. Let $\mathbf{y}(t) = (\mathbf{h} \star \mathbf{x})(t)$. Then

$$\begin{aligned} R_{\mathbf{yx}}(\tau) &\triangleq E\big(\mathbf{y}(nT + \tau)\,\mathbf{x}^H(n)\big) \\ &= T \sum_k \mathbf{h}(\tau + nT - kT)\,E\big(\mathbf{x}(k)\,\mathbf{x}^H(n)\big) \\ &= T \sum_m \mathbf{h}(\tau - mT)\,R_{\mathbf{x}}(m) \\ &= (\mathbf{h} \star R_{\mathbf{x}})(\tau), \end{aligned} \tag{9}$$

the Fourier transform of which is $\mathrm{S}_{\mathbf{yx}}(f) = \mathbf{H}(f)\,T\mathrm{S}_{\mathbf{x}}(fT)$.

B. The Spectra Produced by Periodic Drive Signals

A purely deterministic analysis would suffice here, but a statistical approach is taken for consistency with the next section.

Let M be the period of control input $x(n)$, and for $k = 0\ldots M-1$ let coset $k + M\mathbb{Z}$ or any of its elements index the kth of any M rows or columns. A sum $\sum_i$ over such an index i means a sum $\sum_{i\in\mathbb{Z}/M\mathbb{Z}}$ over the M distinct cosets, and such an index i in an exponent refers to an arbitrary coset element. Any other use of a coset i implies a sum, so that $\delta(f - if_0)$, for example, means $\sum_{\ell\in i}\delta(f - \ell f_0)$. (A sum $\sum_k$ over time index k simply means $\sum_{k=-\infty}^{\infty}$.) It is exactly equivalent to use $0\ldots M$–1 to number rows and columns and to then interpret a sum $\sum_i$ over either as $\sum_{i=0}^{M-1}$. A row or column index *expr* then means *expr* $\bmod M$, and $\delta(f - if_0)$ means $\sum_{\ell=-\infty}^{\infty}\delta(f - (i + M\ell)f_0)$.

If control input $x(n)$ has period M then so does address vector $\mathbf{x}(n)$, so for this periodic analysis let us replace the general code matrix $\mathbf{A}$ depicted in Table 1 with some $\mathbf{A}_p$ customized to this problem as illustrated in Fig. 2, where columns and their labels have been transferred from the original table and ordered so that their column labels now constitute one period of $\mathbf{x}(n)$ in reverse (convenient later) order. Then $y(t) = \mathbf{g}(t) \star \mathbf{A}_p\,\mathbf{u}(n)$ with new vector $\mathbf{u}(n)$ cycling repeatedly through $\widehat{e}_{-1}\ldots\widehat{e}_{-M}$.

Let $\mathcal{E}_n^i$ denote the event $\big[\mathbf{u}(n) = \widehat{e}_i\big]$ so that

$$\begin{aligned} R_{\mathbf{u}}(k) &= E\big(\mathbf{u}(n)\big)\,\mathbf{u}^T(n-k)\,\big) \\ &= \sum_{i,j} \widehat{e}_i\,\widehat{e}_j^{\,T}\,P(\mathcal{E}_{n-k}^j\,|\,\mathcal{E}_n^i)\,P(\mathcal{E}_n^i). \end{aligned}$$

When $k \in j - i$ using coset arithmetic (or $k = j - i \mod M$ using modulo-M indices), $P(\mathcal{E}_{n-k}^j\,|\,\mathcal{E}_n^i) = 1$. It is zero otherwise, so the double sum reduces to a single sum. To eliminate the remaining n dependence and make the autocorrelation stationary, set $P(\mathcal{E}_n^i) = 1/M$ to assert probabilistically that the phase of the periodic cycling is unknown. Then the above reduces to

$$R_{\mathbf{u}}(k) = \frac{1}{M}\sum_i \widehat{e}_i\,\widehat{e}_{i+k}^{\,T}. \tag{10}$$

Matrix $R_{\mathbf{u}}(k)$ in (10) is cyclic for each k and therefore diagonalized by an M-point DFT matrix $\mathbf{D}$ with $\mathbf{D}_{m,n} = \mathrm{e}^{-\mathrm{j}2\pi nm/M}$:

$$\begin{aligned}[\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H]_{m,n} &= \frac{1}{M}\sum_i \mathbf{D}_{m,i}\,\mathbf{D}^*_{n,i+k} \\ &= \frac{1}{M}\sum_i \mathrm{e}^{-\mathrm{j}2\pi(mi-n(i+k))/M} \\ &= \mathrm{e}^{\mathrm{j}2\pi\frac{nk}{M}}\,\frac{1}{M}\sum_i \mathrm{e}^{-\mathrm{j}2\pi\frac{mi}{M}}\left(\mathrm{e}^{-\mathrm{j}2\pi\frac{ni}{M}}\right)^* \\ &= \mathrm{e}^{\mathrm{j}2\pi\frac{nk}{M}}\,\frac{1}{M}[\mathbf{D}\mathbf{D}^H]_{m,n}.\end{aligned}$$

But $\frac{1}{M}\mathbf{D}^H$ represents an inverse DFT, so $\frac{1}{M}\mathbf{D}\mathbf{D}^H = \mathbf{I}$ and $\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H$ is diagonal with $[\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H]_{i,i} = \mathrm{e}^{\mathrm{j}2\pi ik/M}$.

Getting the scale factors right in the Fourier transform $S_{\mathbf{u}}(fT)$ is worth some risk of belaboring the obvious. By definition,

$$S_{\mathbf{u}}(fT) = \sum_k R_{\mathbf{u}}(k)\,\mathrm{e}^{-\mathrm{j}2\pi kfT} \tag{11}$$

$$= \int \sum_k R_{\mathbf{u}}(k)\,\delta(t-kT)\,\mathrm{e}^{-\mathrm{j}2\pi ft}dt, \tag{12}$$

so we can certainly write

$$\begin{aligned}&\left[\mathbf{D}S_{\mathbf{u}}\big((f+\tfrac{i}{MT})T\big)\mathbf{D}^H\right]_{i,i} \\ &= \int\sum_k \left[\mathbf{D}R_{\mathbf{u}}(k)\mathbf{D}^H\right]_{i,i}\mathrm{e}^{-\mathrm{j}2\pi\frac{i}{MT}t}\,\delta(t-kT)\,\mathrm{e}^{-\mathrm{j}2\pi ft}dt \\ &= \int\sum_k \delta(t-kT)\,\mathrm{e}^{-\mathrm{j}2\pi ft}dt,\end{aligned}$$

and the area of the DC line of $\left[\mathbf{D}S_{\mathbf{u}}\big((f+\frac{i}{MT})T\big)\mathbf{D}^H\right]_{i,i}$ or the $f=\frac{i}{MT}$ line of $\left[\mathbf{D}S_{\mathbf{u}}(fT)\mathbf{D}^H\right]_{i,i}$ is the average value of $\sum_k \delta(t-kT)$, or just $\frac{1}{T}$. Using $\frac{1}{M}\mathbf{D}^H\mathbf{D} = \mathbf{I}$ then,

$$TS_{\mathbf{u}}(fT) = \frac{1}{M^2}\mathbf{D}^H\mathbf{\Lambda}(f)\,\mathbf{D},$$

where diagonal matrix function $\mathbf{\Lambda}(f)$ has period $1/T$ and

$$\mathbf{\Lambda}(f) = \begin{bmatrix}\delta(f) & & & \\ & \delta(f-\frac{1}{MT}) & & \\ & & \ddots & \\ & & & \delta(f-\frac{M-1}{MT})\end{bmatrix}$$

for $0 \le f < \frac{1}{T}$. The autocorrelation $R_{\mathbf{u}}(k)$ of vector $\mathbf{u}(n))$ is stationary, so the output power spectrum is, from (4) and (8),

$$\begin{aligned}\mathrm{S}_y(f) &= \mathbf{G}(f)\,\mathbf{A}_p\,TS_{\mathbf{u}}(fT)\;\mathbf{A}_p^H\,\mathbf{G}^H(f) \\ &= \frac{1}{M^2}\left(\mathbf{G}(f)\mathbf{A}_p\mathbf{D}^H\right)\,\mathbf{\Lambda}(f)\,\left(\mathbf{G}(f)\mathbf{A}_p\mathbf{D}^H\right)^H \\ &= \sum_i \delta(f-\tfrac{i}{MT})\left|\mathbf{G}(f)\,\frac{1}{M}\left[\mathbf{A}_p\mathbf{D}^H\right]_{\text{column } i}\right|^2.\end{aligned}$$

The discrete-time drive has period M and so yields an output spectrum of period $1/T$ with each period containing M equally spaced lines. In the above sum, there are M period $1/T$ subfamilies of lines, each containing one line per period. Each subfamily has associated with it a transfer function that determines its lines' areas. The M transfer functions corresponding to the M subfamilies are the M elements of row vector $\frac{1}{M}\mathbf{G}(f)\mathbf{A}_p\mathbf{D}^H$.

C. *The Spectrum Produced by a Random Input*

Let $\mathcal{E}_n^i$ denote the event $\left[\mathsf{N}(\mathbf{x}(n)) = \widehat{e}_i\right]$ so that

$$R_{\mathsf{N}}(k) = \sum_{i,j}\widehat{e}_i\,\widehat{e}_j^T P(\mathcal{E}_{n-k}^j \cap \mathcal{E}_n^i).$$

When $N=4$, $n=9$, and $k=1$ for example, the Fig. 1 delay line maps $x(n)$ to elements of vectors $\mathbf{x}(n)$ and $\mathbf{x}(n-k)$ like this:

element of $\mathbf{x}(9)$:				0	1	2	3
$\ldots\ x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$	$x(8)$	$x(9)\ \ldots$	
element of $\mathbf{x}(8)$:		0	1	2	3		

The first three elements of $\mathbf{x}(9)$ are necessarily also the last three elements of $\mathbf{x}(8)$. Generalizing, event $\mathcal{E}_{n-k}^j \cap \mathcal{E}_n^i$ can be nonnull only when the two subvectors

elements $\max(0,-k)$ through $\min(N-1, N-1-k)$ of $\mathbf{x}_i$
elements $\max(0,k)$ through $\min(N-1, N-1+k)$ of $\mathbf{x}_j$,

are equal, where binary vector $\mathbf{x}_i$ is one of the possible values of address vector $\mathbf{x}(n)$ and is defined by $\mathbf{2}^T\mathbf{x}_i = i$. Event $\mathcal{E}_{n-k}^j \cap \mathcal{E}_n^i$ then corresponds to a specific $\min(2N, N+|k|)$ bits of control input $x(n)$ matching particular values found in various elements of $\mathbf{x}_i$ and $\mathbf{x}_j$. If random control input $x(n)$ has independent and equiprobably binary-valued samples, then

$$[R_{\mathsf{N}}(k)]_{ij} = \begin{cases}\dfrac{1}{2^{\min(2N,N+|k|)}} & \text{for } (i,j,k) \text{ yielding equal subvectors,} \\ 0 & \text{otherwise.}\end{cases}$$

When $|k| \ge N$ the subvectors have length zero and so are automatically equal, and the 2^{-2N} value given here is valid.

When else are the subvectors equal? We have four ways to express each of the two events of interest:

$\mathcal{E}_n^i$	$\mathcal{E}_{n-k}^j$
$\left[\mathsf{N}(\mathbf{x}(n)) = \widehat{e}_i\right]$	$\left[\mathsf{N}(\mathbf{x}(n-k)) = \widehat{e}_j\right]$
$\left[\mathbf{x}(n) = \mathbf{x}_i\right]$	$\left[\mathbf{x}(n-k) = \mathbf{x}_j\right]$
$\left[\mathbf{2}^T\mathbf{x}(n) = i\right]$	$\left[\mathbf{2}^T\mathbf{x}(n-k) = j\right]$

The subvector condition above used the third of these, but there is advantage now in using the fourth instead to map each possible binary address vector to an integer. The matching-subvector condition above becomes the requirement that, if $M(\ell,m)$ and $L(\ell,m)$ respectively denote the m most- and least-significant bits of the address vector that maps to integer ℓ,

for $-N < k < 0$, $\quad M(j, N+k) = L(i, N+k)$;
for $\quad 0 < k < N$, $\quad L(j, N-k) = M(i, N-k)$;
for $\quad k = 0$, $\quad$ either condition above or just $i = j$.

For $0 \le k \le N$ we can write i and j in terms of their bits in common and their other bits as

$$\begin{aligned} j &= 2^{N-k} j_o + b_c \text{ with } 0 \le b_c < 2^{N-k} \\ i &= 2^k \quad b_c + i_o \text{ with } 0 \le i_o < 2^k\end{aligned}$$

so that

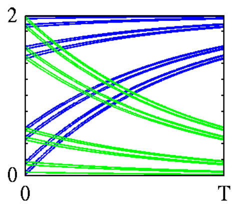
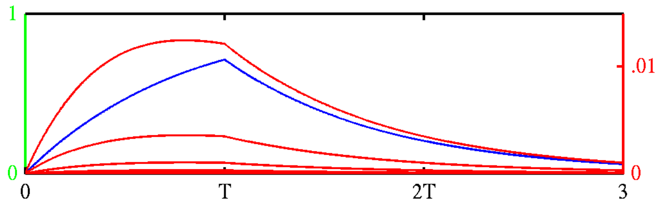


Fig. 3. NRZ data RC filtered with a data-dependent R of $0.8T/C \times \sqrt{1.1}$ (data high) and $0.8T/C/\sqrt{1.1}$ (data low). This 10% time-constant switching yields the worst spectrum in Fig. 4. Left: eye pattern for $\{0, 2\}$ data. Below: elements $\mathbf{g}_1(t)$ (point-peaked curve, left scale), $\mathbf{g}_2(t), \mathbf{g}_3(t), \ldots$ (other curves, right scale).



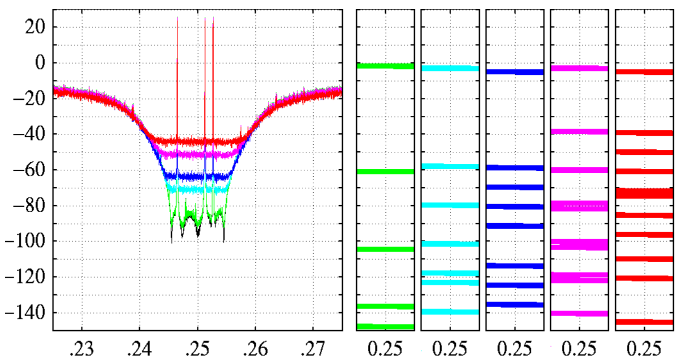


Fig. 4. Large plot on left: bandpass $\Delta\Sigma$ spectrum $T^{-1}S_y(f)$ for three signal tones in a signal band with width 1% of clock rate $1/T$. Each of five plots on right: signal-band levels of $T^{-1}\mathbf{G}_i(f)$ for $i = 1, 2, \ldots$ The bottom of the six curves on the left (almost buried under the next curve) has identical data-high and data-low RC time constants $\tau_{\text{hi}} = \tau_{\text{low}} = 0.2T$. For the other five curves on the left, bottom to top, and the five plots on the right, left to right:

$\sqrt{\tau_{hi}\,\tau_{low}}$	$0.2T$	$0.4T$	$0.8T$	$0.4T$	$0.8T$
$(\tau_{hi} - \tau_{low})/\tau_{low}$	1%	1%	1%	10%	10%

$$R_{\mathsf{N}}(k) = \frac{1}{2^{\min(2N,N+k)}} \sum_{b_c=0}^{2^{N-k}-1} \left(\sum_{i_o=0}^{2^k-1} \widehat{e}_i \right) \left(\sum_{j_o=0}^{2^k-1} \widehat{e}_j \right)^T$$

For $k > N$ the value of $R_{\mathsf{N}}(k)$ is constant, fixed at $R_{\mathsf{N}}(N)$. For negative k, either replace k on the right above with $-k$ and swap i and j or use the conjugate symmetry $R(k) = R^H(-k)$ required of all autocorrelations to obtain $R_{\mathsf{N}}(k)$ from its value for positive k. These yield identical results.

In the $N = 2$ case, for example, $R_{\mathsf{N}}(k)$ is

$$\begin{array}{ccccc} k \le 2 & k = -1 & k = 0 & k = 1 & k \ge 2 \\ \frac{1}{16}\begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} & \frac{1}{8}\begin{bmatrix} 1&1&0&0\\0&0&1&1\\1&1&0&0\\0&0&1&1 \end{bmatrix} & \frac{1}{4}\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} & \frac{1}{8}\begin{bmatrix} 1&0&1&0\\1&0&1&0\\0&1&0&1\\0&1&0&1 \end{bmatrix} & \frac{1}{16}\begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} \end{array}$$

For each k the matrix elements sum to unity, because they are probabilities of events that partition the sample space.

Codeword sequence $\mathbf{a}(n)$ of (3) has autocorrelation

$$R_{\mathbf{a}}(k) = \mathbf{A} R_{\mathsf{N}}(k) \mathbf{A}^H.$$

Using (6) with $\mathbf{C} = \mathbf{C}_{\{0,2\}}$, this effects a remarkable simplification of $R_{\mathsf{N}}(k)$. Not proven here but easily verified computationally for all N likely to be useful in the foreseeable future,

$$R_{\mathbf{a}}(k) = \mathbf{1}\,\mathbf{1}^T + (\mathbf{I} - \widehat{e}_0\,\widehat{e}_0^T)\,\delta_k, \tag{13}$$

where vector $\mathbf{1}$ is $2^{N-1}+1$ ones, $\mathbf{I}$ is a $2^{N-1}+1$ square identity matrix, and here $\widehat{e}_0$ is the length $2^{N-1}+1$ unit vector $[1\ 0\ \cdots\ 0]^T$.

Spectral density $S_{\mathbf{a}}(fT)$ now follows from (12). The first term, $\sum_k \mathbf{1}\,\mathbf{1}^T\,\delta(t - kT)$, is supported on $T\mathbb{Z}$ and has period T, so its Fourier transform has period $1/T$ and is supported on $T^{-1}\mathbb{Z}$ and must be a periodic impulse train. Each area is just that of the DC impulse and so given by the average over t of $\sum_k \mathbf{1}\,\mathbf{1}^T\,\delta(t - kT)$ or just $\frac{1}{T}\mathbf{1}\,\mathbf{1}^T$. Direct evaluation of (12) yields the Fourier transform of $\sum_k (\mathbf{I} - \widehat{e}_0\,\widehat{e}_0^T)\,\delta_k\,\delta(t - kT)$, so periodically extending $\delta(f)$ to $\delta_{1/T}(f) \triangleq \sum_\ell \delta(f - \ell/T)$,

$$T S_{\mathbf{a}}(fT) = \delta_{1/T}(f)\,\mathbf{1}\,\mathbf{1}^T + T(\mathbf{I} - \widehat{e}_0\,\widehat{e}_0^T).$$

Finally, from (8),

$$\begin{aligned} S_y(f) &= \mathbf{G}(f)\,T S_{\mathbf{a}}(fT)\,\mathbf{G}^H(f) \\ &= \delta_{1/T}(f) \left| \sum_i \mathbf{G}_i(f) \right|^2 + T \sum_{i \neq 0} \left| \mathbf{G}_i(f) \right|^2. \end{aligned}$$

The first term contributes clock-harmonic lines, and the second term is white and contributed to by all but the $\mathrm{G}_{\text{clock}}(f)$ component of transformed waveform row vector $\mathbf{G}(f)$.

The fact that the various $|\mathbf{G}_i(f)|^2$ for $i = 2, 3, \ldots$ affect the output with the same scaling suggests that they can be compared to get a rough sense of how the various nonlinear terms will contribute to a system output. This is next explored numerically.

IV. Computation and Conclusions

Suppose a system subjects NRZ data (noninverting unipolar conversion scaled by two) to an RC time constant that is a function of the current data value, giving its output mismatched rise and fall times. Using crosscorrelation (9), the waveform row vector for this system was derived exactly using $\mathbf{A} = \mathbf{C}_{\{0,2\}}\mathbf{X}\mathbf{H}$ in the model. (The details are omitted here for space reasons but will be presented elsewhere.) It is plotted in Fig. 3 (without $\mathbf{g}_0(t)$, as clock harmonics are of little interest in $\Delta\Sigma$ systems) for particular but unremarkable parameters. The huge linear ISI contribution visible in the eye pattern is nearly irrelevant to $\Delta\Sigma$ performance, but the tiny nonlinear-ISI component, visible as slight asymmetry top to bottom and resulting from $\mathbf{g}_2(t)$, raises the noise floor some 40 dB in the associated $\Delta\Sigma$ spectrum in Fig. 4 (left, top curve), completely destroying signal-band SNR. The other Fig. 4 curves suggest—clearly more work is needed here—that $\mathbf{G}_3(f)$ (rather than the larger $\mathbf{G}_2(f)$) dominates the nonlinear spectral contributions. This term contributes much more when time constants are long, so much more mismatch is tolerated when time constants are short.

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