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Synthesis of a Polarization-Controlled Pattern for a Wideband Array by Solving a Second-Order Cone Program

Jeffrey O. Coleman — *jeffrey.coleman@nrl.navy.mil*
Dan P. Scholnik — *scholnik@nrl.navy.mil*
Patrick E. Cahill — *patrick.cahill@nrl.navy.mil*

Naval Research Laboratory (http://www.nrl.navy.mil/)
Radar Division, Signal Processing Theory & Methods Section

***Abstract*— Jointly optimizing complex baseband FIR filters as element weights using second-order cone programming (SOCP) can synthesize a wideband array pattern having an arbitrarily polarized main beam with, in terms of L_1, L_2, L_∞, or other norms, its frequency response flattened and cross-polarization components strictly limited across the entire beam. Sidelobes and SNR measures can be just as flexibly controlled. A high-level interface to a fast solver minimizes required programming.**

Summing the filtered outputs of dual-polarization receive elements with locations in finite set $\mathcal{X}$ yields polarized wideband array pattern, with unit propagation-direction vector $-\widehat{\mathbf{u}}$,

$$\overrightarrow{A}(\widehat{\mathbf{u}}, f) = \sum_{\mathbf{x}\in\mathcal{X}} \left[H_{\nearrow\mathbf{x}}(f)\overrightarrow{F}_{\nearrow\mathbf{x}}(\widehat{\mathbf{u}}, f) + H_{\nwarrow\mathbf{x}}(f)\overrightarrow{F}_{\nwarrow\mathbf{x}}(\widehat{\mathbf{u}}, f)\right] \mathrm{e}^{j2\pi\,\widehat{\mathbf{u}}\cdot\mathbf{x}\,f/c}, \qquad (1)$$

where the two complex vector fields $\overrightarrow{F}_{\times\mathbf{x}}(\widehat{\mathbf{u}}, f)$ are the polarized patterns of the element outputs at location $\mathbf{x}$ and where the filter frequency responses $H_{\times\mathbf{x}}(f)$ act as frequency-dependent element weights. Polarized pattern $\overrightarrow{A}(\widehat{\mathbf{u}}, f)$ and a simpler, unpolarized version have been synthesized using iterative least-squares (ILS) methods [1, 2] and second-order cone progamming (SOCP) [4] respectively. ILS methods are easy to implement in MATLAB, and a high-level interface to efficient solvers [3] now makes the more-general SOCP approach easy also. Here we sketch the latter's extension to the polarized case.

Polarization. Let complex unit vectors $\widehat{\mathbf{p}}_{\circlearrowleft}(\widehat{\mathbf{u}}, f)$ and $\widehat{\mathbf{p}}_{\perp}(\widehat{\mathbf{u}}, f)$ specify co- and cross-polarization respectively, so that $\widehat{\mathbf{u}} \perp \widehat{\mathbf{p}}_{\circlearrowleft} \perp \widehat{\mathbf{p}}_{\perp} \perp \widehat{\mathbf{u}}$ for all f and all $\widehat{\mathbf{u}}$ using inner product $\langle \mathbf{x},\mathbf{y}\rangle \triangleq \mathbf{y}^*\cdot\mathbf{x}$. Conjugate symmetries $\widehat{\mathbf{p}}_{\circlearrowleft}(\widehat{\mathbf{u}}, f) = \widehat{\mathbf{p}}^*_{\circlearrowleft}(-\widehat{\mathbf{u}}, -f)$ and $\widehat{\mathbf{p}}_{\perp}(\widehat{\mathbf{u}}, f) = \widehat{\mathbf{p}}^*_{\perp}(-\widehat{\mathbf{u}}, -f)$ are required. Let scalar co- and cross-polarization patterns $A_{\circlearrowleft}(\widehat{\mathbf{u}}, f)$ and $A_{\perp}(\widehat{\mathbf{u}}, f)$ be related to $\overrightarrow{A}(\widehat{\mathbf{u}}, f)$ of (1) by $A_{\circlearrowleft} = \langle \overrightarrow{A}, \widehat{\mathbf{p}}_{\circlearrowleft}\rangle$ and $A_{\perp} = \langle \overrightarrow{A}, \widehat{\mathbf{p}}_{\perp}\rangle$ respectively.

Receiver Model. Refer all pre-output-sum filtering to the element outputs, prior to any A/D conversion or downconversion, and there write it as $H_{\times\mathbf{x}}(f) \triangleq H_{\text{opt}\times\mathbf{x}}(f)\, H_{\text{fixed}}(f)$ such that $H_{\text{opt}\times\mathbf{x}}(f)$ is linear in the filter weights to be optimized—a key requirement—and $H_{\text{fixed}}(f)$ depends neither on those weights nor on the specific element output.

A Second-Order Cone Program. The *variables* comprise the filter weights and, from below, real variable $\alpha_0 \geq 0$ and complex variables $\alpha_1 \ldots \alpha_N$, all optimized to maximize SNR measure α_0 subject to second-order cone constraints that upper bound nonnegative-definite (NND) quadradic functions of the *variables* by constants or by rank-one NND quadratics in those *variables*. Forms $10^{\gamma/10}$ are γ dB power ratios with illustrative γ's:

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$$\int \sum_{\mathbf{x}\in\mathcal{X}} |H_{\nearrow\mathbf{x}}(f)|^2 + |H_{\nwarrow\mathbf{x}}(f)|^2 \; df_{\text{avg}} \leq 1 \tag{2}$$

$$\left.\begin{aligned} &\int |A_{\circlearrowleft}(\widehat{\mathbf{u}}_n, f) - \alpha_n|^2 \, df_{\text{avg}} \leq 10^{-35/10}\, |\alpha_n|^2 \\ &\int |A_{\perp}(\widehat{\mathbf{u}}_n, f)|^2 \, df_{\text{avg}} \qquad\;\; \leq 10^{-40/10}\, |\alpha_n|^2 \end{aligned}\right\} \left\{\begin{aligned} &\text{for each } n = 0, \ldots, N, \\ &\text{with } \{\widehat{\mathbf{u}}_0, \ldots, \widehat{\mathbf{u}}_N\} \subset \mathcal{U}_{\text{mb}} \end{aligned}\right. \tag{3}$$

$$\|\overrightarrow{A}(\widehat{\mathbf{u}}, f)\|^2 \leq 10^{-10/10} \text{ for each pair } (\widehat{\mathbf{u}}, f) \text{ in a finite subset of } \mathcal{U}_{\text{sl}} \times \mathcal{F} \tag{4}$$

Integrating $df_{\text{avg}} \triangleq 1_{\mathcal{F}}(f)\, df \Big/ \int_{\mathcal{F}} df$ uses $1_{\mathcal{F}}(f) = \left\{\begin{smallmatrix} 1 \text{ if } f\in\mathcal{F} \\ 0 \text{ if } f\notin\mathcal{F} \end{smallmatrix}\right.$ to average over signal band $\mathcal{F}$, so (2) sets output noise to its one-receiver level. In main-beam directions $\{\widehat{\mathbf{u}}_0, \ldots, \widehat{\mathbf{u}}_N\}$ the RMS passband error and gain of frequency responses $A_{\circlearrowleft}$ and $A_{\perp}$ are respectively bounded by (3) relative to nominal main-beam gains $\{\alpha_n\}$, which float except in nominal beam-center direction $\widehat{\mathbf{u}}_0$ because elsewhere only flatness in f matters. Minimizing objective $-\alpha_0$ maximizes the nominal SNR of $10\log_{10}|\alpha_0|^2$ dB. This much synthesizes a basic pattern, but here (4) further improves that pattern by bounding the peak gain, relative to isotropic, to sidelobe inputs of arbitrary polarization using Schwartz inequality $|\langle\overrightarrow{A}, \widehat{\mathbf{p}}\rangle| \leq \|\overrightarrow{A}\|\,\|\widehat{\mathbf{p}}\| = \|\overrightarrow{A}\|$. Integrals are realized as finite Riemann-sum approximations.

Example. A general conformal array above is here simply 21 identical dual-polarized infinitesimal Hertzian dipoles oriented along $(\widehat{\mathbf{x}} \pm \widehat{\mathbf{z}})/\sqrt{2}$ and spaced along $\widehat{\mathbf{x}}$ at intervals $d = \lambda/2$ at 1.45 GHz, the top of 400 MHz signal band $\mathcal{F}$. Co- and cross-polarization vectors $\widehat{\mathbf{p}}_{\circlearrowleft}(\widehat{\mathbf{u}}, f)$ and $\widehat{\mathbf{p}}_{\perp}(\widehat{\mathbf{u}}, f)$ are given by $(\widehat{\boldsymbol{\theta}} \pm \mathcal{H}(f)\widehat{\boldsymbol{\varphi}})/\sqrt{2}$, with unit vectors $\widehat{\mathbf{u}}$, $\widehat{\boldsymbol{\theta}}$, and $\widehat{\boldsymbol{\varphi}}$ mutually orthogonal, with $\widehat{\boldsymbol{\varphi}}$ in the $\widehat{\mathbf{x}}\widehat{\mathbf{y}}$ plane, and with Hilbert-transform response $\mathcal{H}(f) \triangleq \left\{\begin{smallmatrix} -j \text{ if } f>0 \\ j \text{ if } f<0 \end{smallmatrix}\right.$. The latter makes the polarizations circular and orthogonal. Nominal main-beam direction $\widehat{\mathbf{u}}_0$ is $55°$ from boresight $\widehat{\mathbf{y}}$ in the $\widehat{\mathbf{x}}\widehat{\mathbf{y}}$ plane. Peak-constraint region $\mathcal{U}_{\text{sl}}$ is a $\widehat{\mathbf{y}}$-centered hemispheric shell less a main-beam cutout region comprising unit vectors $\widehat{\mathbf{u}}$ with $|(\widehat{\mathbf{u}} - \widehat{\mathbf{u}}_0) \cdot \widehat{\mathbf{x}}| < \sin 9°$. Here $H_{\text{fixed}}(f) = 1$, so $H_{\nearrow\!\!\!\nwarrow\mathbf{x}}(f) = H_{\text{opt}\nearrow\!\!\!\nwarrow\mathbf{x}}(f)$.

For comparison, a reference pattern in Fig. 1 uses weighted ideal time-delay filters $H_{\nearrow\!\!\!\nwarrow nd\widehat{\mathbf{x}}}(f) = \big(\mathrm{Re}[p_{\nearrow\!\!\!\nwarrow}] + \mathrm{sgn}(f)\mathrm{Im}[p_{\nearrow\!\!\!\nwarrow}]\big)\, a_n\, \mathrm{e}^{-j2\pi f n\tau}$ with $\tau = -\widehat{\mathbf{u}}_0 \cdot \widehat{\mathbf{x}}\, d/c$ and Taylor-like array-factor weights $\{a_n\}$ that keep sidelobe peaks 20 dB or more below the peak. Complex polarization-mixing constants $p_{\nearrow}$ and $p_{\nwarrow}$ fix $A_{\perp}(\widehat{\mathbf{u}}_0, f) = 0$ and $A_{\circlearrowleft}(\widehat{\mathbf{u}}_0, f) \neq 0$.

Figure 2 results from solving the SOCP above to jointly optimize all 42 of the $H_{\text{opt}\nearrow\!\!\!\nwarrow\mathbf{x}}(f)$ responses, each modeling a complex baseband FIR digital filter with impulse-response support $\left\{\frac{k}{500\text{ MHz}} : k = -3 \ldots 3\right\}$. Constants in dB are as in (2) through (4), and $\widehat{\mathbf{u}}_1 \ldots \widehat{\mathbf{u}}_{10}$ are symmetric about $\widehat{\mathbf{u}}_0$ at $\widehat{\mathbf{u}}_n \cdot \widehat{\mathbf{x}}$ increments of $\sin 0.5°$. Each of the two single-polarization array factors $\sum_{\mathbf{x}\in\mathcal{X}} H_{\nearrow\!\!\!\nwarrow\mathbf{x}}(f)\, \mathrm{e}^{-j2\pi\mathbf{k}\cdot\mathbf{x}}$, as a function of $\mathbf{k}$ and f, is periodic with the 1×500 MHz rectangular period shown. The element patterns are not periodic, however, so neither are co- and cross-polarization patterns $A_{\circlearrowleft}(\widehat{\mathbf{u}}, f)$ and $A_{\perp}(\widehat{\mathbf{u}}, f)$, which are only defined where $\mathbf{k} = -\widehat{\mathbf{u}} f/c$. The small number of elements used here made peak constraints (4) active across much of the stopband. In a larger array a -10 dBi peak level would make these constraints active only near the main beam.

Conclusion. In an example design, using FIR filters as element weights has resulted in polarization control and frequency-response flatness across the main beam in Fig. 2 that is significantly better than in the Fig. 1 reference pattern. Although available space does not permit it to be demonstrated here, in fact much longer FIR filters would be needed to get comparable sidelobe performance by simply approximating time delays.

Second-order cone programming optimizes a small array's filter weights nicely. For a larger array, a simpler design omitting (4) could be solved as a generalized eigenproblem.

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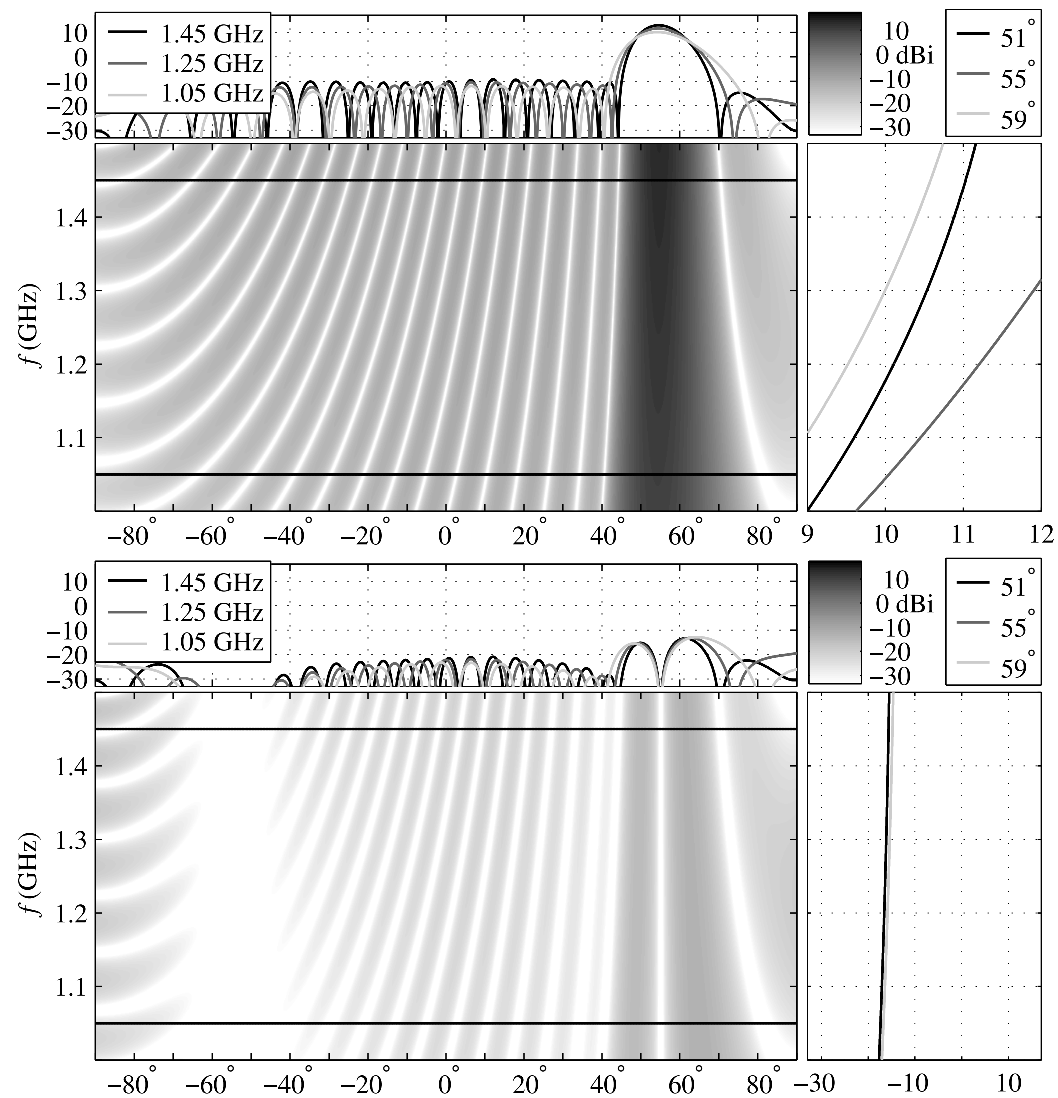

Figure 1: A 21-element dual-polarization $\times$ linear array spaced at $\lambda/2$ at the 1.45 GHz upper band edge is given this conventional reference pattern by combining elements of like polarization with Taylor-like weights, time-delay steering the two single-polarization patterns to $55°$, and baseband combining the two outputs with complex weights to fix beam-center responses in desired circular (top) and undesired orthogonal (bottom) polarizations. Poor polarization control and main-beam frequency-response flatness result off beam center.

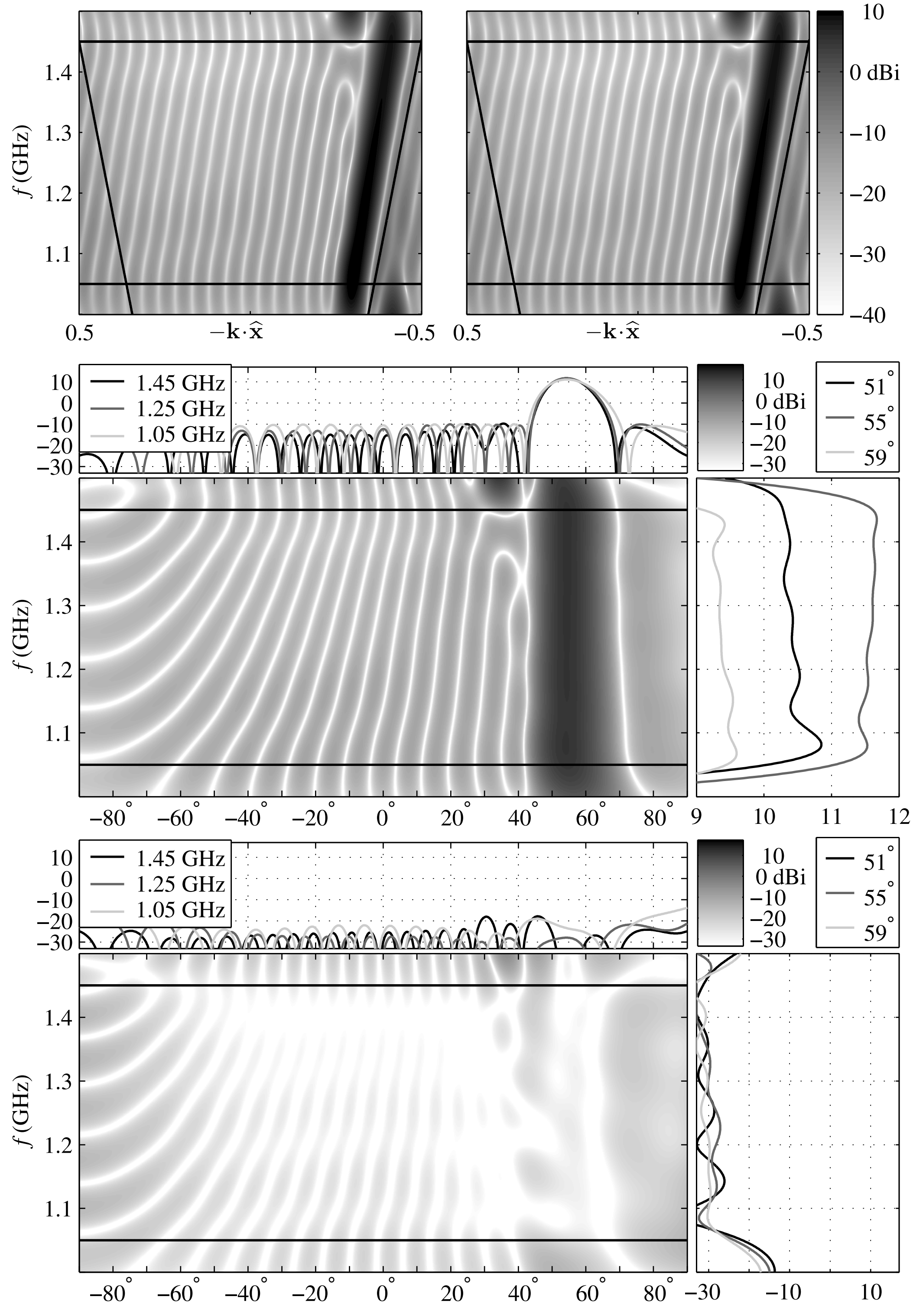

Figure 2: Joint SOCP optimization of 42 seven-tap complex baseband FIR filters used as element weights for a 21-element dual-polarization linear array yields array factors (top) for the two linear $\times$ element polarizations. In f and the component of spatial frequency k along the array the "visible" region $\|\mathbf{k}\| < |f|/c$ is a trapezoid. In the combined, overall-array responses to the desired circular (center) and undesired orthogonal polarizations (bottom) that trapezoid becomes instead a full-width rectangle in f and the angle of k to boresight.