'Incorrect corrections' by ancient editors – a challenge in Chinese mathematical philology

Submitted for a celebration of Guo Shuchun's 80th birthday.

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Abstract: Corruptions of ancient scientific texts stem not only from banal scribal errors, but also from mistaken emendations by well-intentioned early editors. This article considers mathematical examples in three Chinese texts of the Song–Yuan period: *Hefang tongyi* 河防通议, *Mengxi bitan* 梦溪笔谈, and *Shushu jiuzhang* 数书九章.

For Professor Guo Shuchun – Thank you for your friendship, and thank you for your years of research and many publications.

Scholars have long been aware of *scribal errors* in ancient texts, and generations of philologists have developed sophisticated ways of dealing with them. In mathematical texts in particular, the mathematical context is very often a sure guide in identifying this simple type of error.

But historians of Chinese science and technology must also take seriously the possibility of more complex corruptions of their texts, in which an ancient editor, encountering a text that he does not understand, incorrectly 'corrects' the text to make

'sense' to him. Many ancient Chinese technical texts were difficult to read even in their own time. These became increasingly difficult as the centuries passed between then and now; scribes and editors preparing new editions must often have had difficulties in dealing with them, and most often these later editions are all that we have today. I have pointed out several possible examples of this type of problem in my study of ancient Chinese ferrous metallurgy [Wagner 2008, 51 n. *g*; 52 n. *n*; 217; 274 n. 126; 346].

In this article I take up three examples of 'incorrect corrections' in mathematical texts of the Song and Yuan periods. In dealing with editor-introduced textual errors the proper procedure would seem to be: (1) propose a hypothesis as to the intention of the original text, and argue for its historical plausibility; (2) propose a hypothetical course of events that produced, from this, the text as it now appears, suggest how the editor may have interpreted it, and argue for the historical plausibility of the hypothesis. Both requirements are difficult, and will often be impossible, but in these three cases I believe I am able to give plausible hypotheses to explain obvious errors.

In some earlier studies I have referred to this challenge as that of 'the ignorant editor'. Colleagues and friends, some themselves editors, have objected to this perceived slur, so I now refer neutrally to 'incorrect corrections'.

1. Construction of a canal in Hefang tongyi

Hefang tongyi 河防通議, 'Comprehensive discussion of Yellow River conservancy', was edited by Shakeshi 沙克什 (1278–1351, also called Shansi 贍思), a man of Arabic ancestry employed by the Yuan state in posts concerned with river conservancy. In its present form it consists of six 'sections' (*men* 門), divided into a total of 68 'headings' (*mu* 目). The first five sections concern practical engineering and administration, while

the last, 'Calculation' (*Suanfa men* 算法門), concerns the mathematical techniques needed for this work.[1]

The complex history of the text has been studied by Guo Shuchun (1997).[2] Shakeshi had at hand two versions of a *Hefang tongyi* by Shen Li 沈立, completed shortly after 1048. These he called the 'Directorate version' (*jian ben* 監本） and the 'Kaifeng version' (*Bian ben* 汴本). (The Directorate version had previously been in the possession of the Directorate of Waterways, *Dushui jian* 都水監, of the Jin 金 dynasty.) He writes in his preface that both versions were badly organized and difficult to consult; therefore, 'I have removed redundancies, corrected errors, reduced the number of sections, and organized it in categories.'

Shakeshi's book was completed and printed in 1321. There seems to be no way of knowing whether other editions were printed. It was copied into the great Ming encyclopedia *Yongle dadian* 永樂大典 (completed 1408), and this copy was copied into the Qing collectaneum *Siku quanshu* 四庫全書 (completed 1782). All extant versions are ultimately based on the *Siku quanshu* version; no earlier version is now extant. This version, however, contains many banal scribal errors which are corrected in the *Shoushan'ge congshu* 守山閣叢書 and *Congshu jicheng* 叢書集成 editions. (All three editions are available on-line at www.scribd.com/collections/3809180/.)

Comments in smaller characters are scattered throughout the text, and these occasionally include clues to their origin. Some clearly originate in the Directorate

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[1] The best discussion of the 'Calculation' section that I am aware of is that of Guo Shuchun [1997]. It has also been discussed by Yabuuchi Kiyoshi [1965], Guo Tao [1994], and three others, cited by him, whose publications have not been available to me. None deals with the 'confusion' noted by Guo Shuchun.

[2] Other interpretations of Shakeshi's preface are possible, but here I follow Guo Shuchun.

version, and some – those which explicitly compare the Directorate version with the Kaifeng version – are clearly by Shakeshi. Some refer to events after 1321, and must therefore be by some later editor, perhaps the *Yongle dadian* or *Siku quanshu* editors. The few comments in the 'Calculation' section do not happen to provide such clues, and may originate from any of these three sources.

1.1. Calculations for the construction of a canal

Many of the 27 problems in the 'Calculation' section are quite simple, and several give incorrect methods and answers. The problem we are concerned with here is the last and most complex, which concerns the distribution of work among several groups of workers. The Chinese text is reproduced in Figure 13, and a translation is given in Appendix 1.

The problem concerns the construction of a canal, shown here in Figure 1. One group of labourers is to excavate part of it, *IJKLHEFG*, called the 'cut'. The dimensions and volume of the whole canal are given, together with the volume of the cut. The dimensions of the cut are required.

This is a simplified version of a practical problem in construction administration: the available labour determines the volume to be excavated, and the labourers must be told how far they are to dig, *x* in Figure 1.

The given dimensions are:

$$l = 500 \ bu \ 步 \ (\text{'paces'}) = 2500 \ chi \ 尺 \ (\text{'feet'}) \approx 780 \ \text{metres}$$

$$a_1 = 1040 \ chi$$

$$a_2 = 890 \ chi$$

$$b_1 = 1000 \ chi$$

$$b_2 = 850 \ chi$$

$$d = 1 \ zhang \ 丈 \ = 10 \ chi$$

The text relates the two volumes to numbers of 'labour units' (*gong* 功), which seem to correspond to man-days. By the particular administrative norm invoked in the text, one labour unit corresponds to 40 cubic *chi* of the canal, and the volumes of the canal and the cut are:

$$V = 590{,}625 \text{ labour units} \times 40 \ chi^3/\text{labour unit} = 23{,}625{,}000 \ chi^3$$

$$W = 144{,}450 \text{ labour units} \times 40 \ chi^3/\text{labour unit} = 5{,}778{,}000 \ chi^3$$

The answers given are:

$$x = 120 \ bu$$

$$a_3 = 926 \ chi$$

$$b_3 = 886 \ chi$$

The text does not state explicitly whether the work starts at the western or the eastern end of the canal, but these answers indicate that the cut is at the western end, for they satisfy equations derived by consideration of similar triangles,

$$\frac{a_3 - a_2}{a_1 - a_2} = \frac{b_3 - b_2}{b_1 - b_2} \tag{1}$$

However, the answers appear to be incorrect, for calculation of the volume of the cut from these dimensions gives

$$\begin{aligned} \frac{dx}{4} \left(a_2 + a_3 + b_2 + b_3\right) &= \frac{1}{4} \times 10 \times 600 \times \left(890 + 926 + 850 + 886\right) \\ &= 5{,}328{,}000 \ chi^3 \\ &\neq W = 5{,}778{,}000 \ chi^3 \end{aligned} \tag{2}$$

The text arrives at the given answers using the classical Chinese algebra of polynomials known as *Tianyuan yi* 天元一. Briefly, a column of numbers on the counting board represents what we would call the coefficients of a polynomial equation (see e.g. Mei Rongzhao [1966]; Chemla [1982]; Martzloff [1997, 143–149]; Yabuuchi [1965, 303–304]). The manipulations described in the text result in a column of numbers represented by 'counting rods' (*chou* 籌) on the 'counting board':

15   — ‖‖‖‖

94,500   ‖‖‖ ≡ ‖‖‖‖ ◯ ◯

11,556,000   — ┃ ≡ ‖‖‖ ⊥ ◯ ◯ ◯

which is equivalent to the equation

$$15x^2 + 94{,}500x = 11{,}556{,}000 \tag{3}$$

A root of this equation is found using the ancient Chinese version of Horner's Method (see e.g. Wagner [2017]),

$$x = 120 \ bu$$

The derivation of (3) uses a concept seen several times in the chapter, the *ting* 停, a solid which has the same volume as a given solid, but whose volume is easier to calculate. (I am not aware of any other Chinese mathematical text that uses this word with this meaning.) In this case the *ting* is shown in Figure 2.

The widths at the two ends of the *ting* are calculated:

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1040 + 1000}{2} = 1020 \ chi \tag{4}$$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{890 + 850}{2} = 870 \ chi \tag{5}$$

The rate of change of the width of the *ting* along its length from west to east is then

$$K = \frac{c_1 - c_2}{l} = \frac{1020 - 870 \ chi}{500 \ bu} = 0.3 \ chi/bu \tag{6}$$

Let x = the length of the cut in *bu*. Then the width of the *ting* at the cut is

$$c_3 = Kx + c_2 = 0.3x + 870 \ chi \tag{7}$$

Then, including a conversion of *bu* to *chi*, twice the volume of the cut of the *ting* is

$$dx\,(c_3 + c_1) \times 5\ chi/bu = 2W \quad \textbf{[!?]} \tag{9}$$

$$15x^2 + 94{,}500x = 2W = 11{,}556{,}000\ chi^3 \tag{10}$$

This equation has one positive root, $x = 120\ bu$. The breadth of the *ting* at the cut is then calculated, in a curiously roundabout way:

$$c_3 = 2\frac{W}{dx} - c_1 = 2 \times \frac{5{,}778{,}000\ chi^3}{120\ bu \times 5\frac{chi}{bu} \times 10\ chi} - 1{,}020\ chi = 906\ chi \tag{11}$$

This quantity could have been calculated more simply, using (7):

$$c_3 = \frac{x}{l}\left(c_1 - c_2\right) + c_2 = \frac{120\ bu}{500\ bu} \times \left(1020\ chi - 870\ chi\right) = 906\ chi \tag{12}$$

Calculating further,

$$a_3 = \frac{1}{2}\left(2c_3 + \Delta\right) = 926\ chi \tag{13}$$

$$b_3 = \frac{1}{2}\left(2c_3 - \Delta\right) = 886\ chi \tag{14}$$

where $\Delta = a_1 - b_1 = a_2 - b_2 = 40\ chi$.

Using (13) and (14) requires that the difference between widths is the same, Δ, throughout the length of the canal. If instead a_3 and b_3 had been calculated using (1), this requirement would not have been necessary.

1.2. 'Confusion'

'Attentive readers have undoubtedly been able to see that this reasoning is confused.' [Guo Shuchun, 1997, 229]. Equation (9) is not correct: we should expect c_2 rather than c_1 here.

Guo Shuchun's solution of this confusion assumes that it was the original author who was confused, and that, since (9) and (11) include references to c_1, the cut

proceeded from the eastern end of the canal rather than the western. Correcting equation (9), he arrives at the equations for this situation, corresponding to (9) and (10),

$$dx(2c_1 - Kx) \times 5 \; chi/bu = 2W$$

$$-15x^2 + 102{,}000x = 11{,}556{,}000 \; chi^3$$

and the answers,

$$x \approx 115.25 \; bu$$

$$a_3 \approx 1005.38 \; chi$$

$$b_3 \approx 965.38 \; chi$$

If we follow Guo Shuchun's reasoning, but assume that the cut proceeded from west to east (as I have argued above, equation (1)), rather than east to west, the calculation requires correction of three equations, (9), (10), and (11):

$$dx(c_3 + c_2) \times 5\frac{chi}{bu} = 2W \tag{9'}$$

$$15x^2 + 87{,}000x = 11{,}556{,}000 \tag{10'}$$

$$x \approx 129.917 \; bu$$

$$c_3 = \frac{2W}{dx} - c_2 \approx \frac{11{,}556{,}000 \; chi^3}{129.917 \; bu \times 5\frac{chi}{bu} \times 10 \; chi} - 870 \; chi \approx 909.98 \; chi \tag{11'}$$

Then, using (13) and (14),

$$a_3 = \frac{1}{2}(2c_3 + \Delta) \approx 229.98 \; chi$$

$$b_3 = \frac{1}{2}(2c_3 - \Delta) \approx 189.98 \; chi$$

1.3. An alternative hypothesis

A third possibility, which I ask my friend Guo Shuchun to consider, is that the original text gave a correct calculation, that a scribal error corrupted it, and that a later editor, perhaps Shakeshi himself, attempting to make sense of the corrupt text, corrupted it further.

Under this assumption it is a reasonable inference that the given *answers* have not been corrupted, for they satisfy equation (1). This also indicates that the work proceeded from west to east, as shown in Figures 1 and 2. Then the volume of the cut was $W^* = 5{,}328{,}000$ chi^3 (equation (2)), and therefore the number of work units assigned was given in the original text as $5{,}328{,}000 / 40 = 133{,}200$ rather than the $144{,}450$ of the present text. Therefore, in two places in the text (noted in the translation, Appendix 1), the phrase *da kuo* 大闊, 'larger breadth' (c_1) must be taken to be an error for *xiao kuo* 小闊, 'smaller breadth' (c_2). Then correcting equations (9)–(11) and the given intermediate results gives the following calculation:

$$dx\,(c_3 + c_2) \times 5\ chi/bu = 2W^* \tag{9''}$$
$$15x^2 + 87{,}000x = 10{,}656{,}000\ chi^3 \tag{10''}$$

This has one positive root,

$$x = 120\ bu$$

And

$$c_3 = 2\frac{W^*}{xd} - c_2 = 2 \times \frac{5{,}328{,}000\ chi^3}{120\ bu \times 5\frac{chi}{bu} \times 10\ chi} - 870\ chi = 906\ chi \tag{11''}$$

Finally, using either (1) or the method in the text, (13) and (14),

$$a_3 = 926\ chi$$
$$b_3 = 886\ chi$$

These are the answers given in the text.

How may the text have reached its present state? My hypothesis is that the original text gave the number of work units as 133,200 and gave a calculation equivalent to (9″)–(11″). At some point in its history a scribal error crept in: a substitution of *dakuo* 大闊, 'larger breadth', for *xiaokuo* 小闊, 'smaller breadth', in the statement of (9″). This amounts to changing (9″) to (9).

The editor discovers that the given answers do not satisfy (10):

$$15 \times (120)^2 + 94{,}500 \times 120 = 11{,}556{,}000$$

$$\neq 2 \times 40 \times 133{,}200$$

He therefore changes the number of work units to $144{,}450 = 11{,}556{,}000 / (2 \times 40)$. Now the root of the equation is the given answer, $x = 120$ *bu*.

He then calculates c_3, a_3, and b_3, and discovers that (11″) and (13)–(14) do not result in the given answers. But he finds that subtracting c_1 instead of c_2 in (11″) does result in the given answers. He therefore changes *xiaokuo* to *dakuo* in the statement of (11″), turning it into (11).

1.4. Correct results from an incorrect calculation

The fact that a correct c_3 comes out of a calculation containing three errors has an interesting explanation. From (6) and (9), and for simplicity letting x be measured in *chi* rather than *bu*,

$$2W = dKx^2 + dx\left(c_1 + c_2\right)$$

$$= d\left(c_1 - c_2\right)\frac{x^2}{l} + dx\left(c_1 + c_2\right)$$

Considering similar triangles in the same way as in (1),

$$\frac{c_3 - c_2}{c_1 - c_2} = \frac{x}{l}$$

So that

$$c_1 - c_2 = \frac{l}{x}(c_3 - c_2)$$

$$2W = dx(c_3 - c_2) + dx(c_1 + c_2)$$

$$= dx(c_3 + c_1)$$

Then the calculation (11) gives

$$\frac{2W}{dx} - c_1 = (c_3 + c_1) - c_1 = c_3$$

So whatever volume W^{**} is chosen for W, the solution x^{**} of (9), entered into (11), will give the same value of c_3. This would not be the case if c_3 were calculated using the simpler calculation, (12).

2. Arc measurement in Mengxi bitan

Histories of Chinese mathematics generally state that Shen Gua 沈括 (1031–1095) in his book of jottings *Mengxi bitan* 夢溪筆談 ('Dream Brook essays')[3] gave this approximation for the length of an arc of a circle:

$$s \approx b + \frac{2h^2}{d} \tag{17}$$

where (see Figure 3) h is the sagitta, b is the chord, and d is the diameter of the circle. This is historically plausible, for (17) is equivalent to an approximation for the area of a

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[3] On Shen Gua and his book see especially Sivin [1995]; also Holzman [1958].

circle segment in the *Jiu zhang suanshu* 九章算術 ('Arithmetic in nine chapters',

perhaps 1st century CE),[4]

$$S \approx \frac{bh+h^2}{2} \tag{18}$$

A derivation of (17) from (18) proceeds by observing that the area of the circle section *OAQB*, $\frac{sd}{4}$, is equal to the sum of the areas of the segment *AQB* and the triangle *OAB*. The approximation (17) is also equivalent to a proto-trigonometric formula in the 13th-century calendrical text *Shou shi li* 授時曆 ('Canon of the season-granting system'),[5] and was explicitly used by Zhu Shijie 朱世傑 in a book published in 1303, *Siyuan yujian* 四元玉鉴 (Guo Shuchun et al. [2006, 508–511]; Hoe [1977, 297–298]).

However, this is not precisely the formula given in Shen Gua's text. There is a phrase in the text which must be removed to obtain (17); but a comment in smaller characters includes this phrase and gives a very odd interpretation.

All modern studies of *Mengxi bitan* assume that the comments in smaller characters scattered through the text are by Shen Gua himself, but I shall argue here that at least this comment was added by someone else. I conjecture that the original text, including the elided phrase, gave a more complex formula than (17), that some later edition of the book contained a corrupted version of this formula, and that someone published this corrupted version with a comment that attempted to make sense of it.

2.1. The text

The text in question is in chapter 18, 'Arts' (*Jiyi* 技藝) of *Mengxi bitan*. It is reproduced in Figure 4 from the earliest extant version, dated 1305 [*Yuan kan Mengxi*

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[4] Guo Shuchun [2009, 65–68]; Chemla and Guo [2004, 141; 191–193; 773].

[5] $h^4 + (d^2 - 2sd)h^2 - d^3h + s^2d^2 \approx 0$, *h* being approximated, given *s* and *d*, by Horner's method [Sivin, 2009, 66–67]. A derivation is given by Martzloff [1997, 328–329].

bitan, 1975, 18, 4–7; cf. Hu Daojing [1962, 574–587, § 301]. The raised line, p. 4, line 4, indicates the start of a paragraph; this paragraph continues to the last line of p. 7. None of the versions included in Hu Daojing's critical edition (which are all later than this one) has any important differences from this version.

The paragraph starts with an introduction in a form often seen in Shen Gua's book, with first a statement of what is known or commonly thought on a topic, then the introduction of something new:

> In the arts of calculation, the methods for calculating volumes in [cubic] *chi* 尺 ['feet'], for example . . . [list of geometric forms], are complete for all object forms. There remains the technique for 'volumes with interstices' [*xi ji* 隙積]. . . .   (p. 4, lines 4–6)

The text goes on to give methods for calculating the volumes of several geometric forms, then gives a method for 'volumes with interstices', i.e. stacked spheres or similar objects. This is equivalent to a method for summation of a finite series, but treated as a geometric rather than an algebraic problem.[6]

After this, on page 6, line 5, comes what may originally have been the start of a new paragraph:

> Of methods of measuring *mu* 畝 ['acres', i.e., calculating areas], the square, the round, the crooked, and the straight have been perfected. There remains the technique of 'assembling a circle' [*hui yuan* 會圓]. Since a

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[6] Martzloff [1997, 16 fn. 17] gives a very brief summary of the method. Andréa Bréard [1999, 100–118; 357–360] (note also [1998; 2008]) gives a full translation of the main text of the paragraph and analyzes this first part in detail, but does not deal with the difficulties discussed here. Translations are also given by Fu Zong and Li Lunzu [1974] and Hu Daojing et al. [2008, 531–537]; neither deals with these difficulties.

circular field can be 'broken' [*zhe* 折], it should be possible to assemble [*hui* 會] [the pieces] and restore [*fu* 復] the circle. Among the ancient methods there is only the method of 'splitting the circle in the middle' [? *zhong po yuan* 中破圓] to break it, in which the error can be as much as threefold. I have devised a different technique for breaking and assembling [*zhe hui zhi shu* 折會之術].   (p. 6, lines 5–8)

This passage concerns areas, and has no relation to the preceding text on volumes, so the fact that it is not a separate paragraph (does not start on a new line with the initial character raised) may perhaps be a scribal error. (But note the 'two categories' mentioned further on in the text.) Further, it has no relation to what follows. We should expect an explanation of what 'breaking and assembling' means, and how it is done, but neither breaking nor assembling nor areas are mentioned again. Clearly something has been dropped out of the text here, and there appears to be no way of determining with any certainty what Shen Gua meant by *hui yuan*.

Then, without introduction, follows a method for approximating the length of an arc. See Figure 3: first *b* is calculated, given *d* and *h*, using the Pythagorean theorem:

Lay out the diameter [*d*] of the circular field and halve it; let this be the hypotenuse [of a right triangle]. Then from the halved diameter subtract [*jian* 減] 'the value of the cut' [*suo ge shu* 所割數, i.e. the sagitta, *h*], and let the difference be the leg [*gu* 股, the longer leg of the triangle]. Multiply

each by itself and subtract [*chu* 除 !][7] the [squared] leg from the [squared] hypotenuse. Extract the square root [*kaifang chu* 開方除] of the difference to make the base [*gou* 勾, the shorter leg of the triangle]. Double this to make the 'direct diameter' [*zhi jing* 直徑, i.e. the chord, *b*] of the 'cut field' [*ge tian* 割田, the circle segment].   (p. 6, line 8 – p. 7, line 1)

This calculation is

$$b = 2\sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{d}{2} - h\right)^2} \tag{19}$$

which is correct. Then the length of the arc is calculated:

Multiply the 'value of the cut' [*h*] by itself, **shift one place** [*tui yi wei* 退一位, i.e., divide by 10], and double it. Then divide [*chu* 除] the result by the diameter [*d*] and add the 'direct diameter' [*b*] to make the arc [*s*] of the 'cut field'.   (p. 7, lines 1–3)

If one chooses to ignore the very odd 'shift one place', this calculation is equivalent to (17). After this follows a statement whose meaning is not clear, but may perhaps be a reference to some process of successive approximations ([Bréard [1999, 100–118; 357–360; note also [1998; 2008]):

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[7] Shortly before this, the word used for 'subtract' is *jian* 減. *Chu* 除 is occasionally seen in classical Chinese mathematical texts, as here, with the meaning 'subtract', but its more usual mathematical meanings are 'divide' and 'extract a root'. It is a surprise to see the word used with the meaning 'subtract' here, since it is used twice shortly after, respectively with the meanings 'extract a square root' and 'divide'.

If it is cut again [*zai ge* 再割], [the calculation] is the same. Subtracting the previous 'value of the cut' [*h*] gives the 'value of the second cut' [*zai ge zhi shu* 再割之數].   (p. 7, lines 3–4)

Then there is a comment in smaller characters which will be translated and discussed directly below. The text in large characters then concludes:

These two categories are precise techniques which the ancient writers did not reach. My idle ambition lies in this.   (p. 7, lines 9–10)

'These two categories' may be 'volumes with interstices' and 'assembling a circle', or the phrase may refer to something in a missing part of the original text.

2.1.1. The comment

The comment gives a concrete example, with *d* = 10 *bu* and *h* = 2 *bu*. First the chord *b* is calculated from *d* and *h*:

Suppose there is a circular field with diameter [*d* =] 10 *bu*, and one wishes to cut [*ge* 割] [*h* =] 2 *bu*. Letting the halved diameter be the hypotenuse, 5 *bu*, and multiplying this by itself gives 25. Subtracting the amount cut, [*h* =] 2 *bu*, from the halved diameter, letting the difference, 3 *bu*, be the leg, and multiplying this by itself, gives 9. Subtracting this from [the square] outside the hypotenuse [*xian wai* 弦外, i.e., the square on the hypotenuse, 25 *bu*^{2}], one has 16. Extracting the square root gives 4 *bu*, which is the base. Doubling this makes [*b* = 8 *bu* =] the 'direct diameter' of the cut [the chord of the segment].   (p. 7, lines 4–6)

This calculation follows (19) above,

$$b = 2\sqrt{\left(\frac{10\,bu}{2}\right)^2 - \left(\frac{10\,bu}{2} - 2\,bu\right)^2} = 8\,bu$$

So far there have been no difficulties, but from here on the comment is very difficult to explain:

> Multiplying the 'value of the cut' [$h =$] 2 *bu*, by itself gives 4, and doubling this gives 8. **Shifting upward one place** [*tui shang yi wei* 退上一位[8]] gives 4 *chi* 尺. (p. 7, lines 6–7)

The calculation described here gives:

$$\frac{2 \times (2\, bu)^2}{10} = 0.8\, bu^2$$

but the result is stated to be 4 *chi*. It is likely that this value was arrived at in an attempt to convert square *bu* to square *chi* by multiplying by 5 *chi*/*bu* instead of the correct 25 *chi²*/*bu²*.

The rest is mere nonsense:

> This [4 *chi*] is to be divided by the diameter [*d*], but in this case the diameter, 10 [*bu*], is an excessive value [*ying shu* 盈數], and it is not possible to divide, so one simply uses 4 *chi*. Adding this to the 'direct diameter' [*b*] gives the arc [*s*] of the cut [the circle segment]. One obtains in all the diameter of the circle [*yuan jing* 圓徑, *sic*! i.e. the arc of the segment, *s* ≈], 8 *bu* 4 *chi*. (p. 7, lines 7–8)

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[8] The text has *bei* 倍, 'double, multiple', which, following Hu Daojing [1962, 575], I take to be a scribal error for *wei* 位. The characters are graphically similar, the comment refers directly to a parallel sentence in the main text with *wei*, and the result of the calculation appears in fact to be a division by 10.

The commentator seems to believe that, in a division, if the divisor is greater than the dividend, the quotient equals the dividend. This erroneous calculation fortuitously gives the same result as using (17) would give:

$$s \approx b + \frac{2h^2}{d} = 8\ bu + \frac{2 \times (2\ bu)^2}{10\ bu} = 8.8\ bu = 8\ bu + 4\ chi$$

The comment concludes:

> If one cuts again, this method is also followed. If the diameter is 20 *bu*, to calculate the value of the arc, one should halve it and then, as stated, 'divide by the diameter of the circle'.   (p. 7, lines 8–9)

What this might mean is not at all clear to me, and I suspect that it may be further nonsense.

2.2. A hypothesis

It is unlikely that the astronomer and polymath Shen Gua wrote the strange comment translated here. It is more plausible that a later editor wrote it in order to make a kind of sense of a corrupted version of an original text by Shen Gua.

The most common assumption is that the original text gave the formula (17), and that the corruption consisted of the insertion of the phrase 'shift one place'. The comment then attempts to make sense of the corrupted text. Some strange corruptions have occurred in ancient texts, but the insertion of an entirely irrelevant phrase, not found elsewhere in the book, is surely not a very probable scribal error.

I shall suggest another hypothesis to explain Shen Gua's text. The extant part of the text is explicitly a calculation of the length of an arc, and a possible explanation of the problematic phrase 'shift one place' is that it was originally part of a more complex formula. I propose that this formula may have been equivalent to

$$s \approx b + \frac{2h^2}{d} + 0.2h \tag{20}$$

which is (17) with the addition of the term 0.2*h*.[9] This is a much better approximation. See Figure 5: using (17), the maximum error is 5.42%; using (20), the maximum error is 1.86%, and for most of the range of *h* the error is less than 1%.

An ancient Chinese mathematical writer could have expressed multiplication by 0.2 in a number of ways, but one obvious way would be to write 'shift one place and double it', and this exact phrase does in fact occur in the text: *tui yi wei bei zhi* 退一位倍之. It is therefore plausible that Shen Gua's original formula might have been equivalent to (20).

There is no historical evidence that a formula like (20) was ever used in ancient China (or anywhere else), and this is a serious argument against the hypothesis. Nevertheless, it was not a difficult formula to discover.

Using modern software it was of course simple to graph the absolute error of (17) against *h* and observe that the curve lies close to a straight line with slope –0.2 (see Figure 6). Would and could Shen Gua have sought and found the same fact?

First, it is interesting to note that Zhu Shijie 朱世傑 in 1303 used an improvement of (18), the formula in *Jiuzhang suanshu* for the area of a circle segment, by the addition of a corrective term.[10] It is plausible, therefore, that Shen Gua, a bit more than two centuries before this, may similarly have been interested in improving the related approximation (17).

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[9] This is reminiscent of Shen Gua's use, in the first part, of a known formula plus a corrective term to obtain a new result. Bréard 1998: 116; 1999: 153; 2008: 82.

[10] $S \approx \frac{1}{2}h(h+b) + \frac{(\pi-3)b^2}{8}$, which Zhu Shijie uses with two different values of π (Guo Shuchun et al. [2006, 594–597]; Hoe [1977, 295–296; 1978: 149]; Martzloff [1997, 327 (note typographical error)]). The added term is an exact expression for the error of the *Jiuzhang suanshu* approximation in the case of a semicircle, $b = \frac{h}{2}$.

Chinese astronomers were accustomed to fitting linear, quadratic, and cubic relations to empirical data; in fact Shen Gua appears to mention such an interpolation in one of his jottings.[11] If he had sufficient data on the lengths of arcs in relation to chords and sagittae he would have been able to discover (3) quite easily. Such data could have been acquired empirically, for example by directly measuring arcs of a large circular object: a cartwheel 1 metre in diameter would have allowed sufficient precision. Or Shen Gua could have calculated the lengths of several arcs to any desired precision using Liu Hui's method of inscribed polygons (Guo Shuchun [2009, 64–66]; Chemla and Guo [2004, 148–149; 193].

The mention of 'cutting again' in Shen Gua's text suggests that the original text might in some way have been concerned with successive approximations: the same phrase is used by Liu Hui in his calculation of π (Guo Shuchun [2009, 45–53]; Chemla and Guo [2004, 145–148; 176–184]. In that case it is important to note that if Shen Gua used (20) in, for example, a calculation of π by successive approximations, he would not have obtained good results. As can be seen in Figure 5, for very small arcs the error using (20) is much larger than the error using (17).

3. *The area of a banana leaf in* Shushu jiuzhang

The mathematician Qin Jiushao 秦九韶 (ca. 1202–1261) in his *Shushu jiuzhang* 數書九章[12] gives an incorrect and very odd approximation formula for the area of 'a field shaped like a banana leaf'. It seems that hardly anyone, ancient or modern, has attempted to explain the formula. The only attempt to deal with it that I know of is by

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[11] *Yuankan Mengxi bitan* [1975, 7: 19–22]; Hu Daojing [1962: 304–305, § 128]; Hu Daojing et al. [2008, 210–215]; Li Yan [1957, 77]. See also e.g. Qian Baocong [1964, 103–107].

[12] Libbrecht [1973, 2] translates this book title as 'Mathematical treatise in nine sections'.

Qian Baocong [1966, 84–85], described further below. Libbrecht [1973, 108–109] gives a short account of Qian Baocong's suggestion, but goes no further.

These scholars worked long before interactive mathematical software became widely available and made extensive experimentation possible. After a great deal of experimentation I propose below an explanation of Qin Jiushao's formula.

The term *jiaoyetian* 蕉葉田, 'banana leaf field', does not to my knowledge occur anywhere else in extant classical Chinese mathematical texts. Judging from Qin Jiushao's own illustration, seen in Figure 14 below, it seems certain that the term refers to the intersection of two circles of equal radius, Figure 7.

3.1. Qin Jiushao's approximation

Qin Jiushao's text is reproduced in Figure 14 and translated in Appendix 2. His approximation of the area of the 'banana leaf field' extracts the positive root of the quadratic equation (see Figure 7),

$$x^2 + \left[\left(\frac{c}{2} \right)^2 - \left(\frac{b}{2} \right)^2 \right] x = 10(b + c)^3 \tag{21}$$

after which the area approximation is

$$A_{\text{Qin}} = \frac{x}{2} \tag{22}$$

The text gives the full numerical working for a particular case, and from this it is clear that the text of the formula is not corrupt; it is exactly as Qin Jiushao intended, and the text has been understood correctly.

The formula is not at all a good approximation, as we shall see further below, and (21) is dimensionally inconsistent.

3.2. The approximation in the *Jiuzhang suanshu*

The approximation for the area of a circle segment in the *Jiuzhang suanshu,* equation (18) above, gives the area of one-half of the banana leaf, so that an approximation for the area of the banana leaf is

$$A_{JZSS} = \frac{2bc+b^2}{4}$$

3.3. The accuracy of the two approximations

The particular case calculated in the text has $b = 34$ *bu* 步 ('paces') and $c = 576$ *bu*. The result is

$$A_{\text{Qin}} = 10{,}871^{5{,}213}/_{63{,}070} \; bu^2 \approx 10{,}871.1 \; bu^2$$

The approximation of the *Jiuzhang suanshu* gives

$$A_{JZSS} = 10{,}081 \; bu^2$$

So that in this particular case the two approximations are close to each other. The exact value of the area is

$$A = \frac{2bc(b^2 - c^2)+(b^2 + c^2)\sin^{-1}\dfrac{2bc}{b^2 + c^2}}{8b^2} = 13{,}065.1 \; bu^2$$

and the error percentages of the two approximations are in this case respectively 17% and 23%.

Plotting the values of A, A_{Qin}, and A_{JZSS} for $c = 576$ and the full range of b gives the curves shown in Figure 8. It can be seen immediately that the moderate accuracy of A_{Qin} for this particular case is fortuitous. The formula does not in fact give a useful approximation for the area.

Going further, Figure 9 plots the error percentages of the two approximations for various values of c.

3.4. Qian Baocong's modification of Qin Jiushao's formula

Qian Baocong [1966, 84–85] observes that if the constant term in (21) is changed to $\frac{10}{4}\left(\frac{b+c}{2}\right)^4$, a correct result would be obtained in the case $b = c$ (a circle with diameter c) and $\pi \approx \sqrt{10}$. However, Figures 10 and 11 show that the resulting equation,

$$x^2 + \left[\left(\tfrac{c}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2\right] x = \tfrac{5}{32}(b + c)^4$$

$$A_{\text{Qian}} = \tfrac{x}{2}$$

is moderately accurate for $b > 0.4c$, but is not in general a useful approximation.

3.5. A hypothesis

Extensive experimentation with variations on Qin Jiushao's formula has led me to this approximation:

$$x^2 + \left[c^2 - \left(\tfrac{b}{2}\right)^2\right] x = b\left(\tfrac{b}{2} + c\right)^3 \tag{24}$$

$$A_{\text{new}} = \tfrac{x}{2} \tag{25}$$

which can be seen to be similar to (21)–(22). This is a fair approximation, as can be seen in Figure 12.

Note the interesting similarity between Figure 12 and the curves for A and A_{JZSS} in Figure 8. It turned out, to my amazement, that in fact A_{new} is *equivalent* to A_{JZSS}. It can be derived from A_{JZSS} as follows.

$$A \approx A_{JZSS} = \frac{b^2 + 2bc}{4} = \tfrac{1}{2}b\left(\tfrac{b}{2} + c\right)$$

$$A\left(\tfrac{b}{2} + c\right) \approx \tfrac{1}{2}b\left(\tfrac{b}{2} + c\right)^2$$

$$bA + 2cA \approx b\left(\tfrac{b}{2} + c\right)^2$$

Using $2c = b + 2\left(c - \dfrac{b}{2}\right)$,

$$2bA + 2\left(c - \frac{b}{2}\right)A \approx b\left(\frac{b}{2} + c\right)^2$$

Multiplying by $\dfrac{b}{2} + c$,

$$2bA\left(\frac{b}{2} + c\right) + 2\left[c^2 - \left(\frac{b}{2}\right)^2\right]A \approx b\left(\frac{b}{2} + c\right)^3$$

Again using the *Jiuzhang suanshu* approximation, $4A \approx 2b\left(\dfrac{b}{2} + c\right)$,

$$4A^2 + 2\left[c^2 - \left(\frac{b}{2}\right)^2\right]A \approx b\left(\frac{b}{2} + c\right)^3$$

And this is equivalent to (24)–(25).

The algebraic manipulations shown here would not have been impossible for a mathematician of the Song period. Quite another question is why he would have developed this more complicated formula, which gives exactly the same result as the *Jiuzhang suanshu* formula. He may have believed it to be more accurate, or he may simply have wished to 'show off' with a more complicated calculation.[13]

We can imagine that the original text, by a hypothetical mathematician Jia 甲, may have been something like this, equivalent to (24)–(25):

術曰：以長併半廣，再自乘，又廣乘之，為實。半廣、 長各自乘，所得相減，餘為從方，一為從隅，開平方，半之，得積。

Somehow it ended up in Qin Jiushao's book with the first *ban* 半, 'half', moved to a later position and the second *guang* 廣, 'breadth', changed to *shi* 十, 'ten':

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[13] An example of an unnecessarily complicated calculation in Qin Jiushao's book is a formula requiring numerical solution of a tenth-degree polynomial that can be immediately reduced to fifth degree, and is in fact equivalent to a cubic. Bai Shangshu [1966, 296–299]; Libbrecht [1973, 134–140].

術曰：以長併　廣，再自乘，又十乘之，為實。半廣、半長各自乘，所得相減，餘為從方，一為從隅，開平方，半之，得積。

As to the sequence of events by which the first was transformed to the second, numerous scenarios can be imagined. Here is one. There might well have been an expectation that the breadth and height, b and c, would be treated symmetrically, leading a later mathematician or scribe, Yi 乙, to a text that amounts to

$$x^2 + \left[\left(\frac{c}{2} \right)^2 - \left(\frac{b}{2} \right)^2 \right] x = b(b+c)^3$$

$$A_{\mathrm{Yi}} = \frac{x}{2}$$

But when Qin Jiushao (or an intermediate writer, Bing 丙) received this text and applied the calculation to the case $b = 34$ *bu*, $c = 576$ *bu*, he obtained the result $A_{\mathrm{Yi}} = 27{,}878$ *bu*^{2}, which is far from $A_{JZSS} = 10{,}081$ *bu*^{2}. In dealing with this problem he focused, for whatever reason, on the multiplication by the breadth in the linear term of the equation. Experimenting, he found that substituting a constant 10 for the breadth, b, gave the result $A_{\mathrm{Qin}} = 10{,}871$ *bu*, which is close to A_{JZSS}. He therefore emended the text to what we see in Qin Jiushao's book, amounting to the calculation

$$x^2 + \left[\left(\frac{c}{2} \right)^2 - \left(\frac{b}{2} \right)^2 \right] x = 10(b+c)^3$$

$$A_{\mathrm{Qin}} = \frac{x}{2}$$

but did not test the formula for other values of the breadth and length.

4. Closing remarks

If nothing else, I hope I have convinced readers that reference to 'incorrect corrections' may occasionally be necessary when attempting to explain passages in classical Chinese mathematical texts. There will be readers, I am sure, who feel that my

explanations in these three particular cases are too lengthy and convoluted to be convincing. I can only ask them to provide better explanations for the challenges we encounter in these texts.

A reviewer of one of my books makes the accusation that my collaborator and I 'suspect an "ignorant editor" whenever comprehension problems in Chinese syntax arise.' This is not true and not fair, but it does highlight a potential danger. A loose appeal to 'incorrect corrections' can explain away any problem, just as von Däniken's 'ancient astronauts' can explain away the Egyptian pyramids, the Delhi pillar, and much more.[14] To be useful and convincing, such an explanation must include rigorous arguments concerning the text, the mathematics, and the historical plausibility of the two hypotheses: the proposed original text and the series of textual changes that led to the text as we have it today. Readers will judge whether I have lived up to these requirements in the three cases taken up here.

Appendix 1: Translation of the Hefang tongyi *text*

> The *Siku quanshu* text is reproduced in Figure 13. The text includes representations of the setup on the counting board, but these are obviously corrupted and will be ignored here. I have placed philological comments in footnotes and mathematical comments indented in the translation.

> In the following see Figure 1.

Suppose a canal is to be opened. The straight length is [$l =$] 500 *bu*. At the eastern end, the upper breadth is [$a_1 =$] 1,040 *chi* and the lower breadth is [$b_1 =$] 1,000 *chi*. At the

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[14] Erich von Däniken's *Chariots of the gods,* published in 1966, attempted to explain many ancient accomplishments as the work of visitors from outer space.

western end, the upper breadth is [$a_2 =$] 890 *chi*, and the lower breadth [$b_2 =$] is 850 *chi*. The depth is the same [throughout], [$d =$] 1 *zhang*. The total volume is [$V =$] 23,625,000 [cubic] *chi*.

Note that 1 *zhang* 丈 = 2 *bu* 步 = 10 *chi* 尺 ≈ 3.1 metres.

The given total volume of the canal is correct:

$$V = \frac{1}{4} dl\left(a_1 + b_1 + a_2 + b_2\right) = \frac{1}{4} \times 10 \ chi \times 500 \ bu \times 5 \ chi / bu$$

$$\times (1040 + 1000 + 890 + 850 \ chi) = 23,625,000 \ chi^3$$

One labour unit [*gong* 功], when taking earth at 100 *bu*, is 40 [cubic] *chi*, and it is calculated that 590,625 labour units [will be used].

$$\frac{23,625,000 \ chi^3}{40 \ chi^3 / \text{labour unit}} = 590,625 \text{ labour units}$$

It is desired to assign 144,450 labour units. What are the length and breadth of the cut [*jie* 截]?

The 'cut' is *IJKLHEFG* in Figure 1.

Here my hypothesis suggests that the original text had 133,200 work units, and a later editor changed this to 144,450.

Answer: The length of the cut is [$x =$] 120 *bu* and the breadth of the cut is [$c_3 =$] 906 *chi*.

The dimension c_3 is shown in Figure 2.

(The upper breadth of the cut is [$a_3 =$] 926 *chi*, and the lower breadth of the cut is [$b_3 =$] 886 *chi*.)[15]

Method: Lay out the upper and lower breadths at the eastern end [a_1, b_1], add them together, and halve, obtaining [$c_1 =$] 1020 *chi*, which is the larger breadth of the *ting* 停.

The *ting* is shown in Figure 2.

Further lay out the upper and lower breadths at the western end [a_2, b_2], add them together, and halve, obtaining [$c_2 =$] 870 *chi*, which is the smaller breadth of the *ting*. Subtract this from the larger breadth of the *ting*; the remainder, 150 *chi*, is the difference between the breadths. Divide this by the straight length, [$l =$] 500 *bu*, obtaining [$K =$] 3 *cun*, which is the difference per *bu*.

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1040 + 1000}{2} = 1020 \; chi$$

$$c_2 = \frac{a_2 + b_2}{2} = \frac{890 + 850}{2} = 870 \; chi$$

$$K = \frac{c_1 - c_2}{l} = \frac{1020 - 870 \; chi}{500 \; bu} = 0.3 \; chi / bu$$

Let the *tianyuan* 天元 be [$x =$] the length of the cut.[16]

This corresponds to letting the length of the cut be the unknown in a polynomial equation.

Multiply by the difference per *bu* [K]; this is the difference in breadths at the place where the cut stops.

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[15] Comment in smaller characters in the text.

[16] Excising one occurrence of *tian* 天.

$$Kx = c_3 - c_2$$

Add the smaller breadth of the *ting* [c_2]; this is [$c_3 =$] the breadth of the *ting* at the place of the cut.[17]

$$c_3 = Kx + c_2 = 0.3x + 870 \ chi$$

Add the larger breadth [c_1] of the *ting*; these are the breadths at the two ends of the cut of the *ting*.[18]

$$c_3 + c_1 = Kx + c_1 + c_2 = 0.3x + 1{,}890 \ chi$$

My hypothesis suggests that *dakuo* 大闊, 'larger breadth' (c_1), is a scribal error for *xiaokuo* 小闊, 'smaller breadth' (c_2), in an earlier version of the text.

Multiply by [$d =$] the depth, 1 *zhang*; this makes twice the volume per *chi*.

$$d(c_3 + c_1) = dKx + d(c_1 + c_2) = 3x + 18{,}900 \ chi^2$$

Multiply by 5 to make the twice the volume per *bu*.

$$d(Kx + c_1 + c_2) \times 5 \ chi/bu = (15 \ chi^3/bu^2)x + 94{,}500 \ chi^3/bu$$

[Move this to the left].[19]

Multiply by the *yuanyi* 元一 [the unknown in the equation], [$x =$] the length of the cut. This makes twice the volume of the cut.

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[17] Ignoring *gong* 共 and adding *kuo* 濶 after *ting* 停.

[18] Reading *jie* 截 for *cang* 藏.

[19] Necessary addition by the translator, see fn. •• below. {this draft, fn. 21}

$$2W = dx(Kx + c_1 + c_2) \times 5\,chi/bu$$
$$= (15\,chi^3/bu^2)x^2 + (94{,}500\,chi^3/bu)x$$

Convert the original labour units [assigned to] the cut to a volume [W] and multiply by 2, obtaining 11,556,000 [cubic] *chi*.[20]

$$2W = 144{,}450 \text{ labour units} \times 40\,chi^3 / \text{ labour unit} \times 2$$
$$= 11{,}556{,}000\,chi^3$$

Combine [*xiang xiao* 相消] this with what was moved to the left,[21] obtaining 11,556,000 [cubic] *chi* as the *shi* 實 [the constant term of the equation], 94,500 [cubic] *chi* [per *bu*] [as the linear coefficient], and 15 as the *zongyu* 從隅 [the quadratic coefficient].

The equation is

$$(15\,chi^3/bu^2)x^2 + (94{,}500\,chi^3/bu)x = 11{,}556{,}000\,chi^3$$

This has one positive root, $x = 120\,bu$.

Extract the square root, obtaining [$x =$] 120 *bu*; this is the length of the cut.
Set up the labour units of the cut and convert to a volume, obtaining [$W =$] 5,778,000 [cubic] *chi*.

$$W = 144{,}450 \text{ labour units} \times 40\,chi^3/\text{labour unit} = 5{,}778{,}000\,chi^3$$

Divide this by the length of the cut [x] converted to *chi*, obtaining 9,630 *chi*.

$$\frac{W}{x} = \frac{5{,}778{,}000\,chi^3}{120\,bu \times 5\,chi/bu} = 9{,}630\,chi^2$$

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[20] Reading *gui* 歸 for *sao* 埽. Cf. the parallel usage in lines 7 and 9 on the same page.

[21] See fn. •• above. {this draft, fn. 19}

Divide this by the depth, [d =] 1 *zhang*. . . .[22] Double this and subtract the larger breadth of the *ting*, [c_1 =] 1,020 *chi*. The remainder, 906 *chi*, is [c_3 =] the breadth of the *ting* at the cut.

$$c_3 = 2d\frac{W}{x} - c_1 = 2 \times \frac{9{,}630\ chi^2}{10\ chi} - 1{,}020\ chi = 906\ chi$$

My hypothesis suggests that an earlier text had here *xiaokuo* 小闊, 'smaller breadth' (c_1), and an editor changed this to *dakuo* 大闊, 'larger breadth' (c_2).

Double this, obtaining 1,812 *chi*. . . .[23] Subtract the difference between the upper and lower breadths, 40 *chi*; halve the remainder, obtaining 886 *chi*; this is the lower breadth of the cut. Add again 40 *chi*, obtaining 926 *chi*; this is the upper breadth of the cut.

$$\Delta = a_3 – b_3 = 40\ chi$$
$$b_3 = \tfrac{1}{2}\,(2c_3 – \Delta) = 886\ chi$$
$$a_3 = b_3 + \Delta = 926\ chi$$

In accordance with what was asked.

Appendix 2. Translation of Qin Jiushao's text

The *Yujiatang congshu* text is reproduced in Figure 14.

In the following see Figure 7.

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[22] Excising *wei ting jie kuo* 為停截濶, 'this is the breadth of the *ting* at the cut', which is not correct.

[23] Excising *bing shang xia jie kuo* 併上下截濶, "add together the upper and lower breadths of the cut'. The result just obtained is this sum.

A field shaped like a banana leaf has central length [$c =$] 576 *bu* 步 and central breadth [$b =$] 34 *bu*. The circumference is not known. What is the area in *mu*?

Answer: The area of the field is 45 *mu* 畝 1 *jiao* 角 11 ($^{5,213}/_{63,070}$) [square] *bu*.[24]

One *mu* is equal to 240 square *bu*, and one *jiao* is 60 square *bu*.

$$45 \; mu \times 240 \; bu^2/mu + 1 \; jiao \times 60 \; bu^2/jiao + 11 \; ^{5,213}/_{63,070} \; bu^2$$
$$\approx 10{,}871.08 \; bu^2$$

Method: Multiply the sum of the breadth [b] and the length [c] twice by itself. Further multiply this by 10 to make the *shi* 實 [the constant term of the quadratic equation to be solved].

$$shi = 10 \, (b+c)^3$$

Halve the breadth [b]; halve the length [c]; multiply each by itself. Subtract the one from the other; this is the *zongfang* 從方 [the linear coefficient].

$$zongfang = \left(\frac{c}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Let the *zongyu* 從隅 [the quadratic coefficient] be 1.

$$zongyu = 1$$

Extract the square root and halve it to obtain the area.

$$x^2 + \left[\left(\frac{c}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right] x = 10(b+c)^3$$

$$A_{\text{Qin}} = \frac{x}{2}$$

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[24] The fraction is printed in smaller characters.

Working: Adding the length, [c =] 576 *bu*, and the breadth, [b =] 34 *bu*, gives 610. Multiplying this twice by itself gives 226,981,000 [cubic] *bu*. Shifting up one position, that is, multiplying by 10, gives 2,269,810,000 [cubic] *bu*, obtaining this number to be the *shi*.

$$shi = (576+34)^3 \times 10 = 2{,}269{,}810{,}000 \ bu^3$$

Setting up the length, [c =] 576, and halving it gives 288. Multiplying this by itself gives 82,944, at the top [of the counting board]. Further setting up the breadth, [b =] 34 *bu*, and halving it gives 17. Multiplying this by itself gives 289. Subtracting this from the top, the difference is 82,655, and this is the *zongfang*.

$$zongfang = \left(\frac{576}{2} \right)^2 - \left(\frac{34}{2} \right)^2 = 82{,}655 \ bu^2$$

Letting the *zongyu* be 1

The equation to be solved numerically is then

$$x^2 + 82{,}655x = 2{,}269{,}810{,}000$$

and extracting the square root gives 21,742 *bu* with a remainder of 10,426 [bu^2].

The numbers on the counting board are now

the integral part of x	21,742
remainder (*shi*)	10,426
zongfang	126,139
zongyu	1

representing the equation

$$y^2 + 126{,}139 \, y = 10{,}426$$

in which $y = x - 21{,}742$ is the fractional part of x.

Entering the *shang sheng yu* 商生隅 into the *fang*, and further adding the [single] rod of the [*zong*]*yu* [*yusuan* 隅算] gives 126,140 as the denominator.

I do not fully understand the terminology here, but clearly the calculation is

denominator = *zongfang* + *zongyu* = 126,139 + 1 = 126,140

and the numerator is the remainder of the *shi*, 10,426. This is an application of Qin Jiushao's usual approximation for the fractional part of a root of a polynomial [Libbrecht 1973, 198]:

If $0 < y < 1$ and

$$P(y) = \sum_{i=0}^{n} p_i y^i = 0$$

then

$$y \approx \frac{-p_0}{\sum_{i=1}^{n} p_i}$$

This is equivalent to the assumption that *P* is approximately linear in the interval (0,1).

So $x \approx 21{,}742\ ^{10{,}426}/_{126140} = 21{,}742.0826542$, which corresponds well to the exact root, 21,742.0826548.

Halving both the remainder and [*shi* =] the area result of the root extraction gives the final result, [$A = x/2 =$] $10{,}871\ ^{5{,}213}/_{63{,}070}$.

Here the intention is to calculate $A = x/2$, but an error creeps in. The correct result is $A = 10{,}871\ ^{5{,}213}/_{126{,}140}$, but the calculation mistakenly halves the denominator as well as the numerator, obtaining $10{,}871\ ^{5{,}213}/_{63{,}070}$.

Dividing by the *mu* factor, 240 [bu^2/mu] and simplifying gives 45 *mu*, 1 *jiao*, 11 $^{5,213}/_{63,070}$ [square] *bu*.

References

All URLs in this article were confirmed in January 2021.

Bai Shangshu 白尚恕, 1966. Qin Jiushao cewang jiu wen zaoshu zhi tantao 秦九韶测望九问造术之探讨 (Qin Jiushao's methodology in nine problems of distant measurement). In: Qian Baocong et. al. (Eds.), Song Yuan shuxue shi lunwenji 宋元数学史论文集 (Essays on the history of mathematics in the Song and Yuan periods, 960–1368 CE). Kexue Chubanshe, Beijing, pp. 290–303.

Bréard, Andrea, 1998. Shen Gua's cuts. Taiwanese journal for philosophy and history of science 10, 141–162.

———, 1999. Re-Kreation eines mathematischen Konzeptes im chinesischen Diskurs: 'Reihen' vom 1. bis 19. Jahrhundert. Franz Steiner Verlag, Stuttgart.

———, 2008. A summation algorithm from 11th century China: Possible relations between structure and argument. In: A. Beckmann, C. Dimitracopoulos and B. Löwe (Eds.), Logic and theory of algorithms. Berlin / Heidelberg: Springer, pp. 77–83. www.springer.com/gp/book/9783540694052

Chemla, Karine, 1982. Étude du livre «Reflets des mesures du cercle sur la mer». Dissertation, University of Paris XIII.

Chemla, Karine, and Guo Shuchun, 2004. Les neuf chapitres: Le classique mathématique de la Chine ancienne et ses commentaires. Dunod, Paris.

Fu Zong 傅宗 and Li Lunzu 李伦祖, 1974. Xijishu he huiyuanshu – Shen Gua «Mengxi bitan» pingzhu yize 隙积术和会圆术—«梦溪笔谈»评注一则 (The methods of 'volumes of interstices' and 'assembling a circle' – note on a jotting of Shen Gua). Xibei Shifan Daxue xuebao (Ziran kexue ban) 西北大学学报（自然科学版） (Journal of Northwestern Normal University [Natural science edition]) 1974.4, 17–22.

Guo Shuchun 郭书春, 1997. «Hefang tongyi – Suanfa men» chutan «河防通议•算法门» 初探 (Notes on the mathematical chapter of Hefang tongyi). Ziran kexue shi yanjiu 自然科学史研究 ('Studies in the history of natural sciences') 16.3, 223–232. English abstract, pp. 231–232.

Guo Shuchun 郭书春 (Ed.), 2009. Jiuzhang suanshu yizhu 九章筭术译注 (critical edition of Jiuzhang suanshu). In: Zhongguo gudai keji mingzhu yizhu congshu 中国古代科技名著译注丛书. Shanghai Guji Chubanshe, Shanghai.

Guo Shuchun 郭书春, Ch'en Tsai Hsin 陈在新, and Guo Jinhai 郭金海, 2006. Jade mirror of the four unknowns, 2 vols. In: Library of Chinese classics, Chinese–English 大中华文库 英汉对照. Liaoning Education Press, Shenyang. 'Translated into Modern Chinese by Guo Shuchun; translated into English by Ch'en Tsai Hsin; revised and supplemented by Guo Jinhai.'

Guo Tao 郭涛, 1994. Shuxue zai gudai shuili gongcheng zhong de yingyong – «Hefang tongyi · Suanfa» de zhushi yu fenxi 数学在古代水利工程中的应用—«河防通议•算法»的注释与分析 (The use of mathematics in ancient water-control engineering: analysis of the mathematical chapter of Hefang tongyi). Nongye kaogu 农业考古 ('Agricultural archaeology') 1994.1, 271–278, 285.

Hoe, John, 1977, Les systèmes d'équations polynômes dans le Siyuan yujian (1303). In: Mémoires de l'Institut des Hautes Études Chinoises, vol. 6. Presses Universitaires de France, Paris.

Hoe, J., 1978, The jade mirror of the four unknowns – some reflections. Mathematical chronicle 7, 125–156.

www.thebookshelf.auckland.ac.nz/docs/Maths/PDF/mathschron007-015.pdf

Holzman, Donald, 1958. Shen Kua and his *Meng-ch'i pi-t'an*. T'oung Pao 46.3/5, 260–292. www.jstor.org/stable/20185477

Hu Daojing 胡道靜 (Ed.), 1962. Mengxi bitan jiaozheng 夢溪筆談校證 (Critical edition of 'Dream Brook essays'). Shanghai. Facsimile reprint, Shanghai Guji Chubanshe, Shanghai, 1987.

Hu Daojing 胡道靜, Jin Liangnian 金良年, Hu Xiaojing 胡小静, Wang Hong 王宏, and Zhao Zheng 赵峥, 2008. Brush talks from Dream Brook, 2 vols. In: Library of Chinese classics, Chinese–English 大中华文库　英汉对照. Sichuan Renmin Chubanshe, Chengdu / Shenzhen. 'Translated into modern Chinese by Hu Daojing, Jin Liangnian and Hu Xiaojing; translated into English by Wang Hong and Zhao Zheng.'

Li Yan 李儼, 1957. Zhong suanjia de neichafa yanjiu 中算家的內插法研究 (Studies of the use of interpolation by Chinese mathematicians). Liaoning Jiaoyu Chubanshe, Beijing.

Libbrecht, Ulrich, 1973. Chinese mathematics in the thirteenth century: The Shu-shu chiu-chang of Ch'in Chiu-shao. In: M.I.T. East Asian science series, vol. 1. M.I.T. Press, Cambridge, Mass. & London. Facsimile reprint, Dover, Mineola, 2005.

Martzloff, Jean-Claude, 1997. A history of Chinese mathematics. Translated by S. S. Wilson. Berlin / Heidelberg, Springer-Verlag. 'Corrected second printing', 2006. Original: Histoire des mathémathiques chinoises, Masson, Paris, 1987.

Mei Rongzhao 梅荣照, 1966, Li Ye ji qi shuxue zhuzuo 李冶及其数学著作 (Li Ye and his mathematical works). In: Qian Baocong 钱宝琮 (Ed.), Song Yuan shuxue shi lunwenji 宋元数学史论文集 (Essays on the history of mathematics in the Song and Yuan periods, 960–1368 CE). Kexue Chubanshe, Beijing, pp. 104–148.

Qian Baocong 钱宝琮, 1964. Zhongguo shuxue shi 中国数学史 (The history of Chinese mathematics). Kexue Chubanshe, Beijing. Facsimile reprint, 1981.

————, 1966. Qin Jiushao «Shushu jiuzhang» yanjiu 秦九韶«数书九章»研究. In: Qian Baocong et al. (Eds.), Song Yuan shuxueshi lunwenji 宋元数学史论文集 (Essays on the history of mathematics in the Song and Yuan periods, 960–1368 CE). Kexue Chubanshe, Beijing, pp. 60–103.

Sivin, Nathan, 1995. Shen Kua. In: Nathan Sivin, Science in ancient China: Researches and reflections. Variorum, Aldershot, pp. 1–53.

————, 2009. Granting the seasons: The Chinese astronomical reform of 1280, with a study of its many dimensions and a translation of its records. In: Sources and studies in the history of mathematics and physical sciences. Springer, New York.

Wagner, Donald B., 2008. Science and civilisation in China, vol. 5, part 11: Ferrous metallurgy. Cambridge University Press, Cambridge.

————, 2017. The classical Chinese version of Horner's method: Technical considerations. donwagner.dk/horner/horner.html

Yabuuchi Kiyoshi 藪内清, 1965. Kabō tsūgi ni tsuite 河防通議について (On the *Hefang tongyi*). Seikatsu bunka kenkyū 生活文化研究 13, 297–304.

Yuankan Mengxi bitan 元刊梦溪笔谈, 1975. (Facsimile reprint of a Yuan-period edition of 'Dream Brook essays'). Wenwu Chubanshe, Beijing.
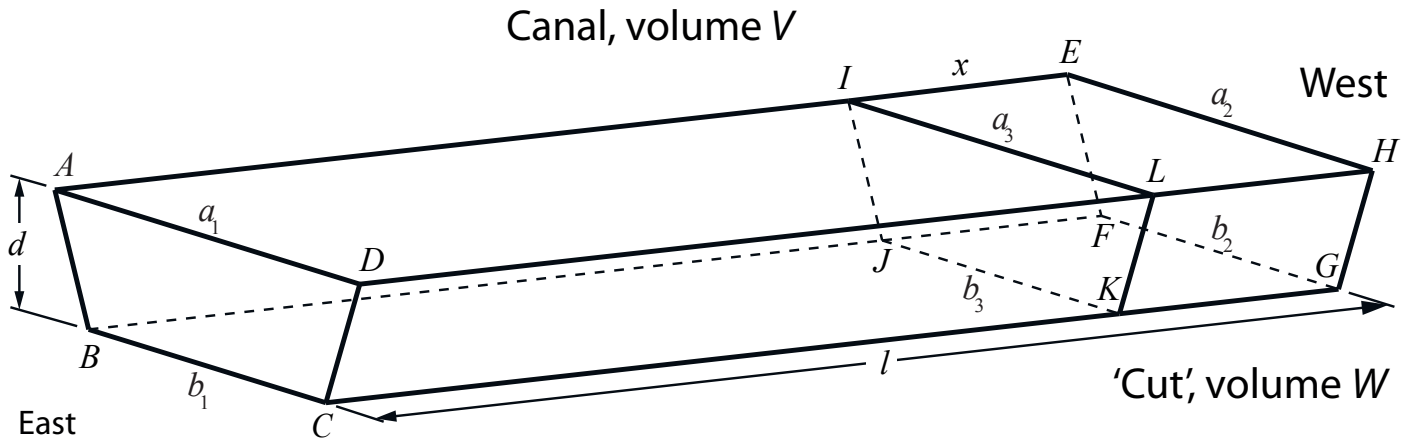
Figure captions
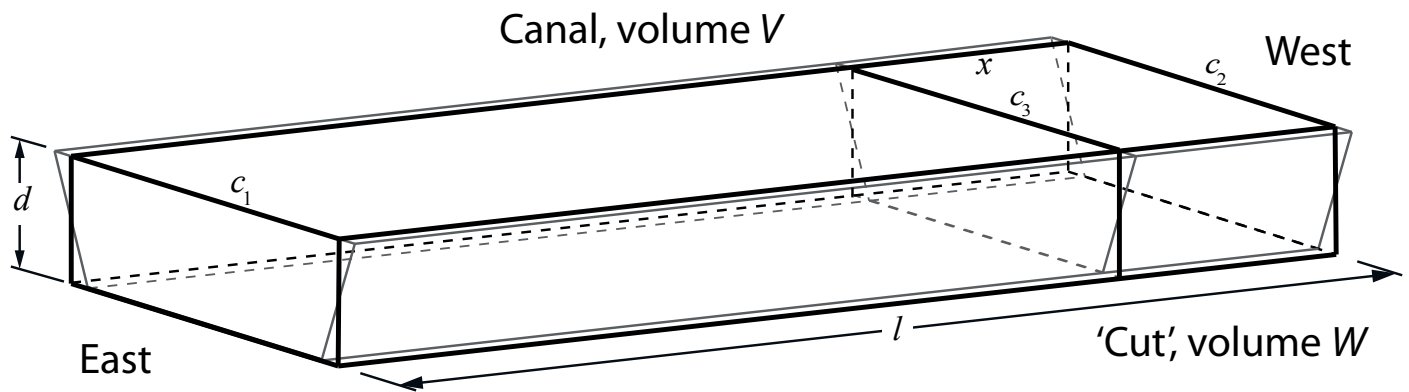
Figure 1. Canal, diagram for Problem 27 of *Hefang tongyi*.

Figure 2. *Ting* 停, geometric construction equivalent to the canal in Figure 1.

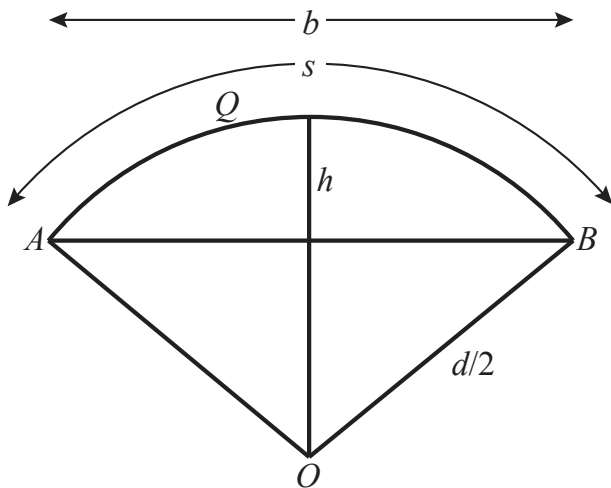Figure 3. Diagram for the calculation in *Mengxi bitan*.

Figure 4. Original text of the *Mengxi bitan* calculation, reproduced from *Yuankan Mengxi bitan* [1975, 18, 4–7].
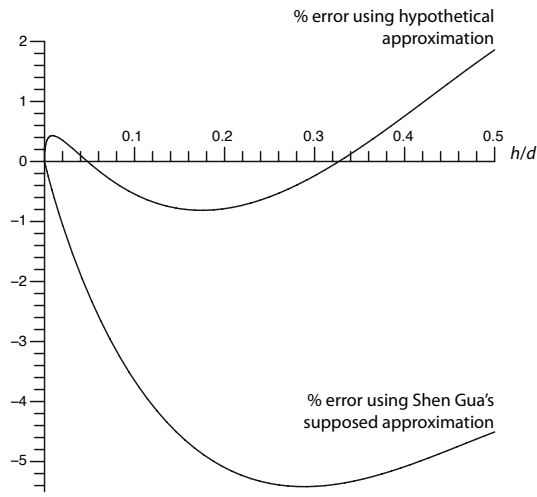
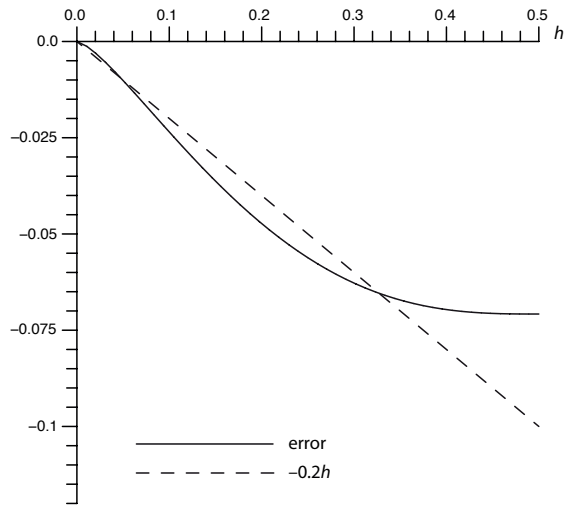Figure 5. Comparison of error percentages in the two approximations (17) and (20).

Figure 6. Absolute error using approximation (17) with $d = 1$ and b calculated from d and h.

Figure 7. Intersection of two circles with equal radii, presumed to be Qin Jiushao's 'banana leaf'.

Figure 8. Comparison of Qin Jiushao's approximation, A_{Qin}, with that of the *Jiuzhang suanshu*, A_{JZSS}, and the exact area of the 'banana leaf field' with $c = 576$ *bu*.

Figure 9. Comparison of the error of the two approximations for various values of c. The curve for A_{JZSS} is the same for all values of c. The error percentage of A_{Qin} is not invariant under scaling because it is not dimensionally consistent.

Figure 10. Comparison of Qian Baocong's modification of Qin Jiushao's approximation, A_{Qian}, with the exact value of the area.

Figure 11. Percentage error of A_{Qian}.

Figure 12. Comparison of the proposed formula, A_{new}, with the exact area of the 'banana leaf field', A, with $c = 576$ *bu*.

Figure 13. Text of Problem 27 in *Hefang tongyi*, reproduced from the *Siku quanshu* 四庫全書 edition, *xia* 下, 24a–25b. www.scribd.com/document/105589337

Figure 14. Text of Qin Jiushao's 'banana leaf' problem, reproduced from the *Yijiatang congshu* 宜稼堂叢書 edition, 5, 14b–15b.
ctext.org/library.pl?if=en&file=83425&page=70

Illustrations for 'Incorrect corrections'
Donald B. Wagner

Canal, volume *V*

West

East

'Cut', volume *W*

Figure 1. Canal, diagram for Problem 27 of *Hefang tongyi*.

Canal, volume *V*

West

East

'Cut', volume *W*

Figure 2. *Ting* 停, geometric construction equivalent to the canal in Figure 1.

Figure 3. Diagram for the calculation in
Mengxi bitan.

文象形,如繩木所用墨斗也。求星辰之行步、氣朔消長,謂之綴術。謂不可以形察,但以筭數綴之而已。北齊祖亘有綴術二卷。筭術求積尺之法,如芻萌、芻童、方池、冥谷、塹堵、鱉臑、圓錐、陽馬之類,物形備矣,獨未有隙積一術。古法:凡筭方積之物,有立方,謂六冪皆方者,其法再自乘則得之。有塹堵,謂如土牆者,兩邊殺,兩頭齊,其法並上下廣,折半以為之廣,以直高乘之,又以直高為句,以上廣減下廣,餘者為股,句股乘弦,

以為斜高。有芻童,謂覆斗者,四面皆殺,其法倍上長加入下長,以上廣乘之,倍下長加入上長,以下廣乘之,并二位,以高乘之,六而一。隙積者,謂積之有隙者,如累棊、層壇及酒家積罌之類,雖似覆斗,四面皆殺,緣有刻缺及虛隙之處,用芻童法求之,常失於數少。余思而得之,用芻童法,為上行、下行別列,下廣以上廣減之,餘者以上廣減之,餘者以高乘之,六而一,併入上行。

相次率至十二,當十二行,一行相次,以先縱橫各二罌上,以芻童法求。假令縱橫各二罌,最下率至十二。

廣之乘倍上行,得長上行,得長十四,併入下廣之乘倍下長,得十六,以上廣之六,併以乘三得,又入上之百之二,上廣得一,併上復覆敲之法,方圓曲直盡矣。未有會圓田既能折之,須使會及三倍者,余別為割數,餘者開方除為句,倍股各自乘,以股除弦,餘者開方除為句,倍。

之為割田之直徑,以所割之數自乘,退一,位倍之,又以圓徑除所得加入直徑為割田之弧,再割亦如之,減去已割之數則再割。之數也,用以假令圓徑十步,欲割二步,以半徑乘半圓徑自乘,得二十五步,以減去平方開之,得股十二步,退股得股。

此二類皆造微之術,古書所不到者,漫志于此。

Figure 5. Comparison of error percentages in the two approximations (17) and (20).

Figure 6. Absolute error using approximation (17) with $d = 1$ and b calculated from d and h.

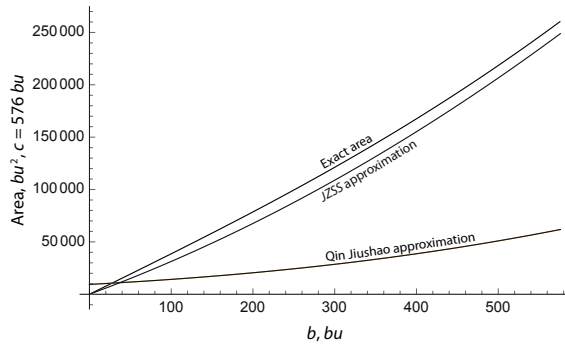Figure 7. Intersection of two circles with equal radii, presumed to be Qin Jiushao's 'banana leaf'.

Figure 8. Comparison of Qin Jiushao's approximation, A_{Qin}, with that of the *Jiuzhang suanshu*, A_{JZSS}, and the exact area of the 'banana leaf field' with $c = 576\ bu$.
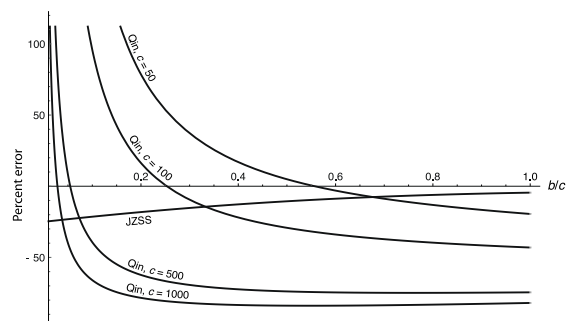
Figure 9. Comparison of the error of the two approximations for various values of c. The curve for A_{JZSS} is the same for all values of c. The error percentage of A_{Qin} is not invariant under scaling because it is not dimensionally consistent.
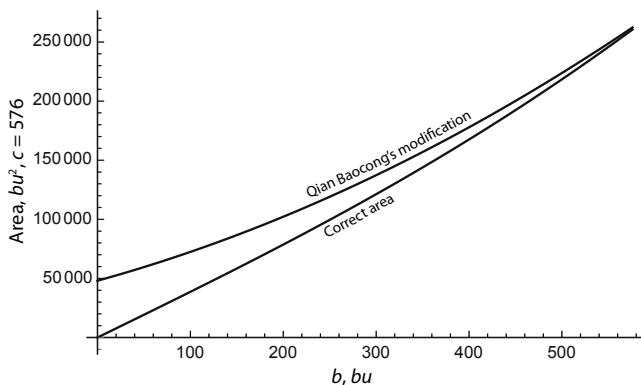

Figure 10. Comparison of Qian Baocong's modification of Qin Jiushao's approximation, A_{Qian}, with the exact value of the area.
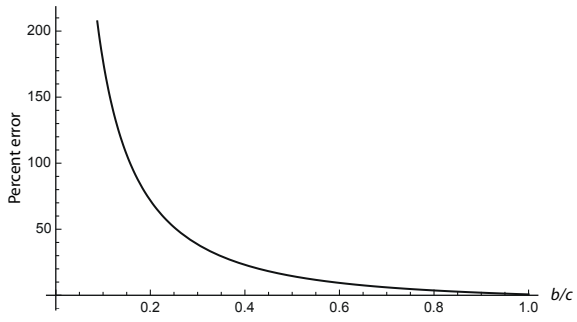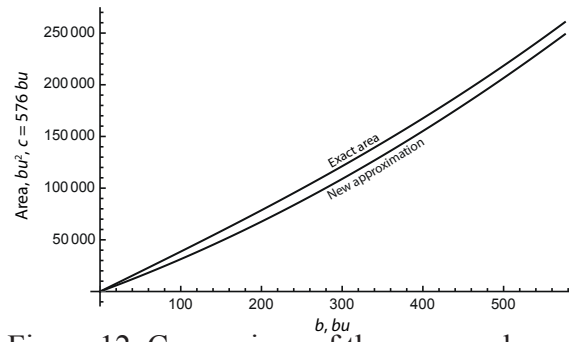
Figure 11. Percentage error of A_{Qian}.

Figure 12. Comparison of the proposed formula, A_{new}, with the exact area of the 'banana leaf field', A, with $c = 576$ *bu*.

每三尺六寸使梢草一束簽椿一條梢草各用中半問

使椿梢多少答曰椿一十二萬條梢草一十二萬束該各

六　法曰高濶相乘得六十四尺以步法五尺通之得

三百二十尺為每步積以長一千三百五十步乘之得

四十三萬二千尺為都積以三尺六寸約之合問

開河

假令開渠河一道正長五百步東頭上濶一千單四十

尺下濶一千尺西頭上濶八百九十尺下濶八百五十

尺同深一丈總積二千三百六十二萬五千尺一百步

取土以四十尺為功計五十九萬單六百二十五功今欲

分一十四萬四千五百功問截長闊多少答曰截（截上濶九百二十六尺截下闊八百十）

長一百二十步截濶九百單六尺

六　法曰置東頭上下濶相併折半得一千單二十尺為

停大濶又置西頭上下濶相併折半得八百七十尺為

停小濶以減停大濶餘一百五十尺為濶差以正長五

百步除之得三寸為每步差立天元為截長一○以

乘每步差為截住處濶差○三加停小濶共為截處停

加停大濶為二段停藏濶○三以深一丈乘之為二

段每尺積五因為二段每步積以元一截長乘之為

二段截積元截功歸積二之得一千一百五十五

萬六千尺與寄左相消得一百五十五萬六千尺為實

九萬四千五百尺從萬一十五從隅平方開之得一百

二十步為截長置截功歸積得五百七十七萬八千尺

以截長歸尺除之得九千六百三十尺以深一丈除之

為停截濶倍之內減停大濶一千單二十尺餘九百單

六尺為停截濶倍之內減得一千八百一十二尺併上下截

濶內減上下濶差四十尺餘半之得八百八十六尺為

截下濶復加四十尺得九百二十六尺為截上濶合問
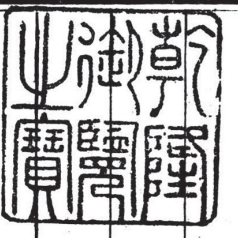
Figure 13. Text of Problem 27 in *Hefang tongyi*, reproduced from the *Siku quanshu* 四庫全書 edition, *xia* 下: 24a–25b.

○蕉田求積

問舊蕉襄田一段中長五百七十六步中廣三十四步不知
其周求積欲合幾何

答曰田積四十五畝一角十一步

術曰以長倂廣再自乘又十乘之爲實半廣半長各自乘
所得相減餘爲從方一爲從隅開平方半之得積

中廣三十四步　中長五百七十六步

草曰以長五百七十六步倂廣三十四步得六百一十以
兩度自乘得二億二千六百九十八萬一千步進一位
即是以十乘之得二十二億六千九百八十一萬步定
得此數以爲實置長五百七十六以半之得二百八十
八自乘得八萬二千九百四十於上又置廣三十四
步以半之得一十七自乘得二百八十九減上餘八萬
二千六百五十五爲從方以一爲從隅開平方得二萬
一千七百四十二步以商生
隅入方又倂隅算其得一十二萬六千一百四十爲母
與不盡及開方田積數皆半之田積定得一萬八千七
十一步六萬三千七百七十分步之五千二百一十三以畝

法二百四十約之得四十五畝一角十一步六萬三
千七十分步之五千二百一十三

數書九章卷五　宜稼堂叢書

Figure 14. Text of Qin Jiushao's 'banana leaf' problem, reproduced from the *Yijiatang congshu* 宜稼堂叢書 edition, 5: 14b–15b.