Mathematics, astronomy, and the planning of public works in China, Han to Yuan (1st century BCE – 14th century CE)

Donald B. Wagner 華道安
Nordic Institute of Asian Studies
University of Copenhagen, Denmark

Home address:
Jernbanegade 9B
3600 Frederikssund
Denmark
www.donwagner.dk

3 March 2022

When I was asked to speak at a conference in 2012 on ‘Civil works, constructions, excavations, and the like’, I realized that I had never before been serious about the *applications* of mathematics in ancient China. When I started looking, I discovered that the subject has some very interesting aspects. One of these is the close relation that appears to have existed between astronomy and public works. I am not aware of any other culture, ancient or modern, in which this relationship is seen.

1. The Nine Chapters

The *Jiuzhang suanshu*, ‘Arithmetic in nine chapters’, is a book of the Han dynasty, perhaps the first century CE. Chapter 5, ‘Calculation of labour’ (*Shang gong* 商功), gives exercise problems which involve the calculation of (1) the volumes of geometric solids and (2) the labour requirements for public works. Historians of mathematics have concentrated their attention on the chapter’s solid geometry, but the labour calculations, though less interesting mathematically, may have been very important to the book’s intended audience. (On the *Jiuzhang suanshu* 九章算術 see especially Chemla & Guo 2004; Loewe 1993: 16–23; Qian Baocong 1963: 83–258.)

An example is Problem 4, which begins (see Figure 1):[1]

> A dyke has breadth 2 *zhang*, upper breadth 8 *chi*, height 4 *chi*, and length 12 *zhang* 7 *chi*. What is the volume?
>
> Answer: 7,112 [cubic] *chi*.
>
> 今有堤下廣二丈，上廣八尺，高四尺，袤一十二丈七尺。問積幾何？
>
> 答曰：七千一百一十二尺。
>
> (In this period 1 *zhang* 丈 = 10 *chi* 尺 ≈ 2.3 metres.]

The *Jiuzhang suanshu* has already given the method for calculating this volume in Problem 1: It is equivalent to

$$V = \frac{(a+b)h}{2}$$

In this case,

$$\frac{(20\,chi\ +\ 8\,chi) \times 4\,chi \times 127\,chi}{2} = 7112\,chi^3$$

Problem 4 continues:

> By the winter norm, one person's labour is 444 [cubic] *chi*. How many labourers are used?
>
> Answer: 16 $^{2}/_{11}$ persons.

[1] Figure 1 near here.

Method: Let the volume in [cubic] *chi* be the dividend and the norm in [cubic] *chi* be the divisor. Divide the dividend by the divisor; [the result] is the number of labourers.

冬程人功四百四十四尺。問：用徒幾何？

答曰：一十六人、一百一十一分人之二。

術曰：以積尺為實，程功尺數為法，實如法而一，即用徒人數。

This calculation is

$$\frac{7112\ chi^3}{444\ chi^3\ /\ \text{labourer}} = 16\tfrac{2}{11}\ \text{labourers}$$

This labour calculation was quite simple, but Problem 21 complicates things. It begins (see Figure 2):[2]

A pool has upper breadth 6 *zhang*, length 8 *zhang*, lower breadth 4 *zhang*, length 6 *zhang*, and depth 2 *zhang*. What is the volume?

Answer: 70,666 $^2/_3$ [cubic] *chi*.

今有盤池，上廣六丈，袤八丈，下廣四丈，袤六丈，深二丈。問積幾何？

答曰：七萬六百六十六尺、太半尺。

The method for calculating this volume has already been given under Problem 18. It is equivalent to

$$V = \frac{[(2l + k)a + (2k + l)b]d}{6}$$

[2] Figure 2 near here.

In this case,

$$\frac{[(2 \times 80\ chi + 60\ chi) \times 60 chi + (2 \times 60\ chi + 80\ chi) \times 40\ chi] \times 20\ chi}{6}$$

$$= 70666\tfrac{2}{3}\ chi^3$$

The problem continues:

> The round trip carrying earth is 70 *bu* 步 [‘paces’], with 20 paces up and down wooden steps. Two [*bu*] on the steps correspond to five on a level path. For resting time, one is added for each 10. Time for loading and unloading is equivalent to 30 *bu*. One round trip is determined to be 140 *bu*. The capacity of a basket of earth is 1 *chi* 6 *cun* [i.e. 1.6 cubic *chi*]. The autumn norm for one person’s labour is equivalent to walking 59 $^1/_2$ *li*.
>
> What is the volume [of earth] carried by each person, and how many labourers are used?
>
> Answer: One person carries 204 [cubic] *chi*, and the number of labourers is 346 $^{62}/_{153}$.
>
> Method: . . .

> 負土往來七十步，其二十步上下棚除。棚除二當平道五，踟躕之間十加一，載輸之間三十步，定一返一百四十步。土籠積一尺六寸，秋程人功行五十九里半。問人到、積尺、用徒各幾何？
>
> 答曰：人到二百四尺。用徒三百四十六人、一百五十三分人之六十二。
>
> 術曰：以一籠積尺乘程行步數為實。往來上下，棚除二當平道五。置定往來步數，十加一，及載輸之間三十步以為法。除之，所得即一人所到尺。以所到約積尺，即用徒人數。

(In this period 1 *li* 里 = 300 *bu* 步 = 1800 *chi* 尺 ≈ 400 metres.)

The method given (not translated here) amounts to

$$\frac{10.6\ chi^3 \times 59\tfrac{1}{2}\ li\ /\ \text{person}\ \times 300\ bu\ /\ li}{\left[(70\ bu - 20\ bu) + \frac{5}{2} \times 20\ bu\right] \times 1.1} = 204\ chi^3\ /\ \text{person}$$

$$\frac{70660\tfrac{2}{3}\ chi^3}{204\ chi^3\ /\ \text{person}} = 346\tfrac{62}{153}\ \text{persons}$$

This calculation is simple arithmetic, and the point of the problem seems to be to exercise the student's ability to organize a complicated calculation.

2. Large labour forces

These labour calculations must remind us of one of the primary characteristics of Chinese governments throughout the ages – their ability to organize large numbers of workers for major projects. One example is that the First Emperor put 700,000 forced labourers to work building his palace and mausoleum; archaeology is beginning to show how it was possible not only to house, feed, and control so many workers, but also to organize their efforts in such a way that useful work was actually accomplished (Wagner 1993: 206).

Jumping two millennia ahead we can see the same phenomenon in Figure 3, an illustration of the digging of a canal in about 1840:[3] This picture is worth some study. At first sight it appears chaotic, but a closer look shows an amazing degree of organization. One can distinguish officials, surveyors, foremen, labourers of various categories, and soldiers guarding the whole.

[3] Figure 3 near here.

3. Xu Shang 許商, 1st century BCE

Xu Shang was the first of a number of astronomers who are known to have been active in the planning of public works. He held various high positions in government (*Han shu*, 19b: 836, 841–842; Loewe 2000: 622), and the *Han shu* bibliography lists two books by him, both long lost:

- *Wuxing zhuanji* 五行傳記 or *Wuxing lun li* 五行論曆, in one chapter, seemingly a commentary concerning natural phenomena and calendrical calculations for the ancient classic *Shang shu* 尚書 (or *Shu jing* 書經, 'Book of documents') (*Han shu*, 30: 1705, 88: 3604).

- *Xu Shang suanshu* 許商算術, in 26 chapters, on calendrical calculations (*Han shu*, 30: 1766).

A story of unfortunate advice by Xu Shang to the Emperor shows the importance of labour calculations like those we have seen above in the *Jiuzhang suanshu* in the administration of the Empire. About 100 BCE the Yellow River had bifurcated, forming a tributary which was called the Tunshi River 屯氏河. In 39 BC this river silted up; seven years later, the chief commandant of Qinghe Commandery 清河郡 (near modern Handan, Hebei), Feng Qun 馮逡, reported that the Yellow River was in danger of overflowing its dykes, and that dredging the Tunshi River would ease the pressure and reduce the danger of flooding.

> The memorial was passed to the Chancellor and the Imperial Counsellor. They responded that Xu Shang was an authority on the *Shang shu*, that he was good at calculating, and that he could estimate the labour required. He was sent to inspect the situation, and reported that the Tunshi River was the cause of flooding, but that the local labour resources were insufficient, and

dredging could be postponed. (*Han shu* 29: 1686; tr. Needham 1971: 329–331)

事下丞相、御史，白博士許商治尚書，善為算，能度功用。遣行視，以為屯氏河盈溢所為，方用度不足，可且勿浚。

The author of the original memorial, Feng Qun, has a very brief biography in the *Han shu* (79: 3305), seemingly only because his family had a certain importance. He held various local administrative posts, rising to Governor of Longxi Commandery 隴西郡 (near modern Lanzhou 兰州, Gansu). We may perhaps surmise that the initiative for public works often came, as in this case, from local officials at lower levels of the bureaucracy who most often are invisible in the extant sources. It was then the task of officials at higher levels to organize a labour force for the project, drawing on the population of more than one locality. This would often involve calculating volumes of earth to be moved; then the number of man-days required could be calculated and compared with the corvée labour available according to prevailing norms. (Very little is known about these norms in the Han period.)

Xu Shang's advice was unfortunate, for three years later the Yellow River overflowed its dykes and caused devastation, flooding 5000 square kilometres to a depth up to 7 metres. Wang Yanshi 王延世 was sent to deal with the problem, and he repaired the dyke with bamboo 'stoppers' (*zhu luo* 竹落, 'gabions'), ca. 9 metres long and 5 metres in circumference, filled with small stones.

An incident two years later shows something of the attitude toward calculation experts. The Yellow River again overflowed its dykes. It was proposed that Wang Yanshi should be sent, but Du Qin 杜欽 objected that an additional expert, Yang Yan 楊焉, should be sent, together with Xu Shang and Shengma Yannian 乘馬延年:

Wang Yanshi and Yang Yan will certainly have violent disputes; there will be deep discussions and mutual criticism. Xu Shang and Shengma Yannian both understand calculation and can estimate labour requirements, so that they can distinguish truth from error. They can choose the correct plan and follow it, so that there surely will be success. (*Han shu* 29: 1686; tr. Needham 1971: 329–331)

延世與焉必相破壞，深論便宜，以相難極。商、延年皆明計算，能商功利，足以分別是非，擇其善而從之，必有成功。

The expedition was a success, and Wang Yannian was given a rich reward.

4. Zu Xuan 祖暅, 5th century CE

Zu Xuan (or Geng, or Gengzhi 之) was a son of the famous mathematician Zu Chongzhi 祖冲之 (429–500 CE), and was himself an important mathematician. He is especially well known for his derivation of the correct formula for the volume of a sphere, using a version of Cavalieri's Theorem (see e.g. Wagner 1978). Less well known is his work on astronomy, for example in the discussion seen in Figure 4 and translated here:[4]

This statement by Jiang Ji 姜岌 is incorrect: 'Stars are like the moon, receiving [light] from the sun and only then being visible.' If stars were inside the sun [?], they would necessarily have phases; this is not the case. It is well known that stars ... have a constant brightness. Thus the bodies of stars have their own light and do not receive [light] from the sun before they

[4] Figure 4 near here.

> shine. ...
>
> (*Kaiyuan zhanjing* 開元占經, *Siku quanshu* 四庫全書 edn, 1: 23b).

A story of unwanted advice by Zu Xuan to the Emperor, and his subsequent imprisonment, indicates that his duties included the supervision of public works. About 500 CE, Wang Zu 王足 presented calculations and proposed that the Huai River 淮水 should be dammed to provide irrigation for Shouyang 壽陽 (perhaps in modern Anhui).

> The Emperor Gaozu 高祖 was in favour of this, and sent the waterworks master Chen Chengbo 陳承伯 and the Construction Supervisor [*caiguan jiangjun* 材官將軍] Zu Xuan to inspect the topography. They both reported that the sandy mud of the Huai River was too light and insubstantial, so that the project could not be completed. (*Liang shu* 梁書, 20: 291–2; *Nan shi* 南史, 45: 1374–5)
>
> 高祖以為然，使水工陳承伯、材官將軍祖晅視地形，咸謂淮內沙土漂輕，不堅實，其功不可就

The Emperor did not accept the experts' advice, and set some 200,000 labourers to work building the dam. It was to be closed in 515, but various difficulties were encountered, the people were distressed, and many labourers died. It was finally closed in 516, and was a great success. Later, however, it was not properly maintained.

> In the autumn, in the 8th month, the Huai River rose disastrously, the dam broke in many places, and the river flowed violently to the sea. Zu Xuan was charged and sent to prison. (*Liang shu* 梁書, 20: 291–2; *Nan shi* 南史, 45: 1374–5)

至其秋八月，淮水暴長，堰悉壞決，奔流于海，祖暅坐下獄

It seems that Zu Xuan, as 'Construction Supervisor', was responsible for the maintenance of the dam, and was punished because he failed in this duty.

5. Wang Xiaotong 王孝通 (6th–7th century CE)

Wang Xiaotong was an important astronomer, known for, among other things, his contribution to a debate on the exact amount of the precession of the equinoxes. He served the Sui and Tang dynasties in posts concerned with calendrical calculations. His book, *Jigu suanjing* 緝古算經, presented to the Imperial Court shortly after 626, contains very little about astronomy or the calendar and a great deal about public works.

(On Wang Xiaotong and his book see e.g. Jiu Tang shu 舊唐書, 32: 1165–1168, 47: 2039, 44:1892; *Xin Tang shu* 新唐書, 25: 534, 27a: 601, 79: 2711–2714, 59: 1547; Qian Baocong 1963: 487; Lim & Wagner 2017.)

The book contains:

- One simple pursuit problem, stated as an astronomical problem.
- 13 problems in solid geometry, stated as construction problems.
- 6 problems in plane geometry, completely abstract.

5.1. The dimensions of a dyke

Problem 3 is as follows (see Figure 5): [5]

> Suppose a dyke is to be built. The difference between the upper and lower widths of the west end is 6 *zhang* 8 *chi* 2 *cun*, the difference between the

[5] Figure 5 near here.

upper and lower widths of the east end is 6 *chi* 2 *cun*, the height of the eastern end is 3 *zhang* 1 *chi* less than the height of the western end, the upper width is 4 *chi* 9 *cun* greater than the height of the eastern end, the straight length is 476 *chi* 9 *cun* greater than the height of the eastern end. (Translation by Tina Su Lyn Lim in Lim and Wagner 2017: 140, 82–84.)

假令築隄，西頭上、下廣差六丈八尺二寸，東頭上、下廣差六尺二寸，東頭高少於西頭高三丈一尺，上廣多東頭高四尺九寸，正袤多於東頭高四百七十六尺九寸。

(In this period 1 *zhang* 丈 = 10 *chi* 尺 = 100 *cun* 寸 ≈ 3 metres)

The given quantities are:

$b_2 – a_2$ = 682 *cun*

$b_1 – a_1$ = 62 *cun*

$h_2 – h_1$ = 310 *cun*

$a – h_1$ = 49 *cun*

$l – h_1$ = 4,769 *cun*

$a = a_2 = a_1$

Labour data, translated directly below, allows the computation of the volume V of the dyke, and its dimensions are found by solving the cubic equation numerically using the Chinese version of Horner's method (Wagner 2017):

$$h_1{}^3 + \left[(l - h_1) + (a - h_1) + \frac{h_2 - h_1}{2} + \frac{b_1 - a_1}{2}\right] h_1{}^2$$

$$+\left[\frac{h_2 - h_1}{2} + \frac{b_1 - a_1}{2} + \frac{a - h_1}{2} + K_4\right] h_1 = V - K_4(l - h_1)$$

where

$$K_4 = \frac{(h_2 - h_1)(a - h_1)}{2} + \frac{(h_2 - h_1)(b_2 - b_1)}{6} + \frac{(h_2 - h_1)(b_1 - a_1)}{2}$$

numerically,

$$h_1{}^3 + 5{,}004\, h_1{}^2 + 1{,}169{,}953\tfrac{1}{3}\, h_1 = 41{,}107{,}188\tfrac{1}{3}\, chi^3$$

$$h_1 = 31\ cun\ \approx 93\ \text{cm}$$

after which the other dimensions are easily found using the given differences:

$h_2 = 341$ *cun* ≈ 10 m

$a_1 = 80$ *cun* ≈ 2.4 m

$a_2 = 80$ *cun* ≈ 2.4 m

$b_1 = 142$ *cun* ≈ 4.3 m

$b_2 = 762$ *cun* ≈ 23 m

$l = 4{,}800$ *cun* ≈ 144 m

Hints in the text indicate that the cubic equation above was derived using a volume dissection something like that shown in Figure 6.[6]

It is clear that this first part of the problem does not reflect actual practice, since no practical construction starts with the *differences* of dimensions. However we shall

[6] Figure 6 near here.

see further below that this same dissection method is used in problems which do appear more practical.

5.2. Building the dyke

Problem 3 continues:

> County A [sends] 6,724 workers, county B 16,677 workers, county C 19,448 workers, and county D 12,781 workers. Each person from the four counties can in one day excavate 9 *dan* 9 *dou* 2 *sheng* [= 9,920 *ge*] of soil. Each person can build a constant volume of 11 *chi* 4 $^{6}/_{13}$ *cun* [i.e. 114 $^{6}/_{13}$ *cun*^{3}] per day. Digging out 1 [cubic] *chi* of soil results in 8 *dou* [= 800 *ge*] of soil.
>
> People in former times, carrying 2 *dou* 4 *sheng* 8 *ge* [= 248 *ge*] of soil on their backs and travelling 192 *bu* on a level road, did 62 trips in one day. In the present situation there are hills to climb and rivers to cross to obtain the soil: there are only 11 *bu* of level road, the slanted height of the hill is 30 *bu*, and the width of the river is 12 *bu*. When climbing a hill 3 [*bu*] is equivalent to 4 [*bu* of level road], when descending a hill 6 [*bu*] is equivalent to 5 [*bu*], and when crossing water 1 [*bu*] is equivalent to 2 [*bu*].
>
> For rest on a level road, one is added for every ten [*bu*]. Loading and unloading [is equivalent to transportation of] 14 *bu*. In the calculations, one man's work is simplified to an equal volume.
>
> The four counties build the dyke together, and the work is completed in one day.
>
> 甲縣六千七百二十四人，乙縣一萬六千六百七十七人，丙縣一萬九千四百四十八人，丁縣一萬二千七百八十一人。四縣每人一日穿土九石九斗二升。每人一日築常積一十一尺四寸、十三分寸之六。穿方一尺

得土八斗。古人負土二斗四升八合，平道行一百九十二步，一日六十二到。今隔山渡水取土，其平道只有一十一步，山斜高三十步，水寬一十二步，上山三當四，下山六當五，水行一當二，平道踟躕十加一，載輸一十四步。減計一人作功為均積，四縣共造，一日役畢。今從東頭與甲，其次與乙、丙、丁。問：給斜、正袤，與高，及下廣，并每人一日自穿、運、築程功，及隄上、下高、廣各幾何？

(In this period 1 *dan* 石 = 10 *dou* 斗 = 100 *sheng* 升 = 1000 *ge* 合 ≈ 60 litres.)

As noted above, from these figures the total volume of the dyke can be calculated, as follows.

Labour for the task of transporting earth:

$$\frac{192\ bu \times 62 \text{ round-trips / man·day}}{\left(11\ bu\ +\ 30\ bu \times \frac{4}{3} + 30\ bu \times \frac{5}{6} + 12\ bu \times 2\right) \times 1.1 + 14\ bu}$$

$$= 96 \text{ round-trips / man·day}$$

$$\frac{248\ ge \text{ / round-trip}}{800\ ge/chi^3} \times 96 \text{ round-trips / man·day} = 29.76\ chi^3 \text{/ man·day}$$

Labour for the three tasks (digging, transporting, and tamping the earth):

$$\frac{29.76\ chi^3 \text{/ man·day}}{1 + \left(\frac{800\ ge/chi^3}{9{,}920\ ge} + \frac{1}{11\frac{29}{65}\ chi^3}\right) \times 29.76\ chi^3} = 4.96\ chi^3 \text{/ man·day}$$

Volume of the dyke:

$$V = 4.96\ chi^3 \text{/ man·day} \times 55{,}630 \text{ men } \times 1 \text{ day} = 2{,}755{,}924.8\ chi^3$$

There has been a tendency to see this kind of complicated problem in simple arithmetic as a gratuitous ‘imbroglio’ (Martzloff 1997: 140), and indeed, just as in the first part of

Problem 3, where *differences* of dimensions are given, this part is far from real practice, for the volume of a dam is not calculated from the amount of labour expended. However, the next part of the problem uses much the same methods, and is much closer to the needs of administrators of public works like Xu Shang and Zu Xuan.

5.3. The contribution of each county

We now encounter a semi-realistic situation in which the labour used is given and the dimensions of a construction are to be calculated. The counties build on the dyke in turn, starting from the eastern end, as seen in the Figure 7. First the volume of County A's contribution is calculated from the number of labourers sent:[7]

$$V_A = 4{,}960\ cun^3 \text{ / man} \times 6{,}724 \text{ men} = 33{,}351{,}040\ cun^3$$

Then the length l_A of the county's contribution is calculated. This is a root of the cubic equation

$$l_A{}^3 + \frac{3b_1 l}{b_2 - b_1} l_A{}^2 + \frac{K_6 l^2}{K_5} l_A = \frac{6V_A l^2}{K_5}$$

where

$$K_5 = (b_2 - b_1)(h_2 - h_1) = 192{,}200\ cun^2$$

$$K_6 = 3(a_1 + b_1)h_1 = 20{,}646\ cun^2$$

Numerically,

$$l_A^3 + 3298\ {}^{2}/_{31}\ l_A^2 + 2474941\ {}^{29}/_{31}\ l_A = 23{,}987{,}761{,}548\ {}^{12}/_{31}\ cun^3$$

$$l_A = 1{,}920\ cun \approx 58 \text{ metres}$$

The other dimensions are calculated by considering similar triangles.

---

7 Figure 7 near here.

The administrator can hereafter mark out the end-point of the construction which County A's labourers must reach before they are finished and may return home. Presumably the detailed organization of this labour is the responsibility of local officers of the county.

The dimensions of the contributions of Counties B, C, and D are calculated by the same method.

The cubic equation given above seems to have been derived by a volume dissection something like that shown in Figures 8 and 9.[8]

6. Textbooks?

Correspondents have suggested to me that books like the *Jiuzhang suanshu* and *Jigu suanjing* could not seriously have functioned as textbooks, for problems like the ones translated here are not practical problems. Even the last problem above, which looks more practical than the others, is not a real-world problem, for real dykes follow terrain rather than being strictly rectilinear. Labour forces might also be expected to be more differentiated – note the many different categories of workers in the 1840 drawing in Figure 3 above.

But this view appears to follow from a misunderstanding of the way textbooks function. Take for example a problem from a modern calculus textbook for engineers (I have unfortunately forgotten which):

[8] Figures 8 and 9 near here, on same page spread.

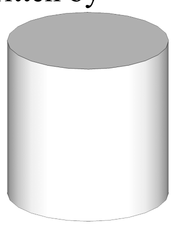
> What relation between the height and the diameter of a tin can minimizes the amount of material used for a given volume?
>
> Answer: height = diameter.[9]

This is not at all a real-world problem: the form of a tin can is determined by all manner of considerations, among which the cost of materials is barely significant. But the techniques used to solve the problem are important for any engineer, and they are best taught using such simple ‘unreal’ problems.

Similarly, it is not reasonable to expect in an ancient Chinese mathematics textbook a realistic description of the planning of an actual dyke or canal, including all of the complications that arise from the particular characteristics of the terrain, the labour force, and much else. But historical sources, some of which have been cited above, suggest that the techniques taught in these books were in fact used by officials responsible for organizing public works.

7. A handbook of river conservancy, 11th–14th century

A book of the Song and Yuan periods shows the sort of calculations which river conservancy officials in this time were expected to carry out. *Hefang tongyi* 河防通議, ‘Comprehensive discussion of Yellow River conservancy’, was originally written by Shen Li 沈立 in the 11th century and extensively edited by Shakeshi 沙克什 in the 14th century. Discussions in the book include the history of the Yellow River, the administrative organs responsible for its control, engineering details concerning the construction of walls, dykes, and canals, and a whole chapter devoted to labour norms for various types of work. A final chapter on ‘calculation’, *Suanfa* 算法, gives 27 calculation problems. (For more on the



---

[9] Figure Q here, to the right of the quotation, with no caption or figure number.

complex history of this book, see Guo Tao 1994; Guo Shuchun 1997; and Wagner 2012.)

7.1. Labour calculations

In the fifth of these problems we see a fairly simple calculation concerning labour:

> Suppose 15,350 bundles of straw [*shaocao* 梢草] are to be carried on foot to a site. The carrying cost has been set to 244 coins per 100 *li* 里 [56 km] and 100 *jin* [60 kg]. Each bundle weighs 15 *jin* [9 kg], and the distance to the site is 90 *li*. How much is the total carrying cost?
>
> Answer: 505 strings [of 1,000 coins] and 629 coins.
>
> Method: Lay out the carrying cost for 100 *li*, 244 coins. Multiply by 90 *li*, obtaining 21 strings and 960 coins. Divide by 100 *li*, obtaining 219.6 coins for 90 *li*, the carrying cost for 100 *jin*. Divide by 100 *jin*, obtaining 2.196 coins, the carrying cost for one *jin*. Multiply [*tong* 通] by 15 *jin* per bundle, obtaining 32.94 [coins], the carrying cost for one bundle. Multiply by the total quantity of straw; the result is the answer.
>
> 假令有稍草一萬五千三百五十束，過腳赴場送納，議定百里百斤腳錢二百四十四文，每束一十五斤，到場九十里，問總該腳錢多少？
>
> 答曰：五百五貫六百二十九文。
>
> 法曰：列百里腳錢二百四十四文，以乘九千里，得二十一貫九百六十文；以百里約之，得九十里腳錢二百一十九文六分，即百斤腳錢也。以百斤約之得二文一分九釐六毫爲一斤腳錢。以每束十五斤通之，得三十二文九分四釐爲一束腳錢，又以總梢草數乘之，得數合問。(守山閣版，下卷，11b–12a).

This calculation is

$$\frac{244 \text{ coins} \times 90\ li}{100\ li \times 100\ jin} \times 15\ jin \text{ / bundle} \times 15{,}350 \text{ bundles} = 505{,}629 \text{ coins}$$

$$= 505 \text{ strings} + 629 \text{ coins}$$

7.2. Volume calculations

Several other problems concern calculations with volumes. The most interesting of these is the last, in which the length of a part of a canal constructed by one group of labourers is to be calculated. See Wagner 2012 for details; briefly, the dimensions and the volume, V, of the whole canal are given. From simple labour data the volume of the 'cut' at the western end, W, can be calculated, and the length of the cut, x, is to be calculated. See Figure 10.[10] The given quantities are:

$V = 23{,}625{,}000\ chi^3$

$W = 5{,}778{,}000\ chi^3$

$l = 500\ bu = 2{,}500\ chi \approx 780$ metres

$a_1 = 1040\ chi$

$a_2 = 890\ chi$

$b_1 = 1{,}000\ chi$

$b_2 = 850\ chi$

$d = 1\ zhang = 10\ chi$

And the length of the cut is a root of the equation

[10] Figure 10 near here.

$$15x^2 + 94{,}500x = 2W = 11{,}556{,}000\ chi^3$$

$$x = 120\ bu$$

No doubt this result could have been arrived at using the sort of volume dissections which we have seen above in the *Jigu suanjing*, but here a more advanced method is used, the algebraic system generally referred to as *tian yuan yi* 天元一. In this system the coefficients of polynomials are represented with rods on the counting board in the same way as had long been done in the Chinese version of Horner's method, and are manipulated in much the same way as we manipulate equations algebraically; see Wagner 2012 for details, and for a discussion of a philological problem that I have ignored here.

8. Concluding remarks

We have seen here two astronomers, Xu Shang and Zu Xuan, whose official duties included the organization of public works. A later example is the famous astronomer Guo Shoujing 郭守敬 (1231–1316). He was one of the leading figures in the major astronomical reform of 1280, but throughout his career he held posts concerning the maintenance of waterways. It may be that the reason for this connection between astronomy and public works is that both fields require the organization of large quantities of data in their calculations. (On Guo Shoujing see Ho Peng Yoke in de Rachewiltz et al. 1993: 282–299; Sivin 2009.)

The three mathematical texts discussed above include many problems concerning, on the one hand, volumes and dimensions of solids, and on the other, labour norms and costs. We may also note the specific mention in the source of Xu Shang's ability to 'estimate the labour required'.

To repeat what has already been suggested above, it is likely that the actual detailed design of a dyke or canal or other large project would have been the responsibility of local officials, but that the overall organization of the project, which would have required workers and materials from more than one jurisdiction, was the responsibility of officials at a higher level of the bureaucracy. It is these higher officials whom we can see in the historical sources. Among their specific tasks were the calculation of volumes of earth and the organization of the workers involved, and these two types of calculation are richly represented in the extant mathematical texts used in their training.

9. References

Bielenstein, Hans. 1980. *The bureaucracy of Han times*. (Cambridge studies in Chinese history, literature and institutions). Cambridge: Cambridge University Press.

Chemla, Karine, and Guo Shuchun. 2004. *Les neuf chapitres: Le classique mathématique de la Chine ancienne et ses commentaires*. Paris: Dunod.

Guo Shuchun 郭书春. 1997. '"Hefang tongyi – Suanfa men" chutan' «河防通议•算法門»初探 (Notes on the mathematical chapter of Hefang tongyi). *Ziran kexue shi yanjiu* 自然科学史研究 ('Studies in the history of natural sciences') 16.3: 223–232. English abstract, pp. 231–232. www.scribd.com/doc/105105547.

Guo Tao 郭涛. 1994. 'Shuxue zai gudai shuili gongcheng zhong de yingyong – «Hefang tongyi · Suanfa» de zhushi yu fenxi' 数学在古代水利工程中的应用—«河防通议•算法»的注释与分析 (The use of mathematics in ancient water-control engineering: analysis of the mathematical chapter of Hefang tongyi). *Nongye kaogu* 农业考古 ('Agricultural archaeology') 1994.1: 271–278, 285. www.scribd.com/doc/138461669.

Han shu. 1962. 漢書. Beijing: Zhonghua Shuju 中華書局. 12 vols.

Hanyu da cidian. 1986–93. 漢語大詞典. Shanghai: Hanyu Da Cidian Chubanshe 漢語大詞典出版社. 12 vols. + index.

Hucker, Charles O. 1985. *A dictionary of official titles in Imperial China*. Stanford: Stanford University Press.

Jiu Tang shu. 1975. 舊唐書. Shanghai: Zhonghua Shuju 中華書局.

Liang shu. 1973. 梁書. Beijing: Zhonghua Shuju 中華書局.

Lim, Tina Su-lyn 林淑玲, and Donald B. Wagner. 2017. The continuation of ancient mathematics: Wang Xiaotong's Jigu suanjing, algebra and geometry in seventh-century China. (NIAS reports 51). Copenhagen: NIAS Press.

Loewe, Michael, ed. 1993. *Early Chinese texts: A bibliographical guide*. Berkeley: Society for the Study of Early China & Institute of East Asian Studies, University of California.

Loewe, Michael. 2000. *A biographical dictionary of the Qin, Former Han and Xin periods (221 BC – AD 24)*. (Handbuch der Orientalistik 60). Leiden, Boston, Köln: Brill.

Martzloff, Jean-Claude. 1997. *A history of Chinese mathematics*. Tr. by Stephen S. Wilson. Berlin / Heidelberg: Springer-Verlag. 'Corrected second printing', 2006. Orig. *Histoire des mathémathiques chinoises*, Paris: Masson, 1987.

Nan shi. 1975. 南史. Beijing: Zhonghua Shuju 中華書局.

Needham, Joseph. 1971. *Science and civilisation in China*. Vol. 4, part 2: *Civil engineering and nautics*. Cambridge: Cambridge University Press.

Qian Baocong 錢寶琮, ed. 1963. *Suanjing shi shu* 算經十書 (Ten mathematical classics). Beijing: Zhonghua Shuju. www.scribd.com/doc/53797787 .

de Rachewiltz, Igor, Hok-lam Chan, Hsiao Ch'i-ch'ing, and Peter W. Geier, eds. 1993. *In the service of the Khan: Eminent personalities of the early Mongol-Yüan period (1200–1300)*. Wiesbaden: Harrassowitz.

Shen Kangshen 沈康身, John N. Crossley, and Anthony W.-C. Lun. 1999. *The nine chapters on the mathematical art: Companion and commentary*. Oxford / Beijing: Oxford University Press / Science Press.

Sivin, Nathan. 2009. *Granting the seasons: The Chinese astronomical reform of 1280, with a study of its many dimensions and a translation of its records* 授時曆叢考. (Sources and studies in the history of mathematics and physical sciences). New York: Springer.

Todd, O. J. 1949. ‘The Yellow River Reharnessed’. *Geographical Review* 39.1: 38–56.

Wagner, Donald B. 1978. ‘Liu Hui and Tsu Keng-chih on the volume of a sphere’. *Chinese science* 3: 59–79. donwagner.dk/SPHERE/SPHERE.html.

——— 1993. *Iron and steel in ancient China*. (Handbuch der Orientalistik, vierte Abteilung: China 9). Leiden: Brill.

——— 2012. ‘Another ignorant editor? Note on a mathematical problem in a 14th-century Chinese treatise on river conservancy’. donwagner.dk/another.htm.

——— 2017. ‘The classical Chinese version of Horner’s method: Technical considerations’. donwagner.dk/horner/horner.html.

Xin Tang shu. 1975. 新唐書. Shanghai: Zhonghua Shuju 中華書局.

[Figure captions]

Figure 1. Dyke, Problem 4 of Chapter 5, *Jiuzhang suanshu*.

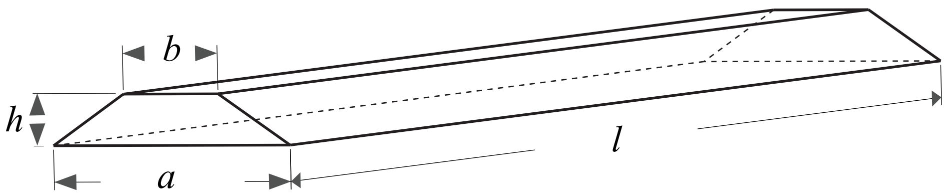


Figure 2. Pool, Problem 21 of Chapter 5, *Jiuzhang suanshu*.

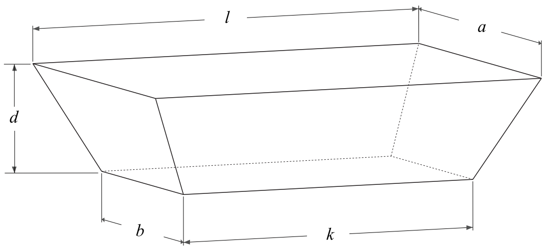


Figure 3. 'Opening a canal', about 1840, copied from Linqing 麟慶, *Hongxue yinyuan tuji* 鴻雪因緣圖記 in Needham 1971: 262.

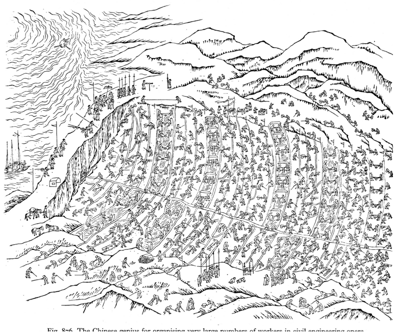
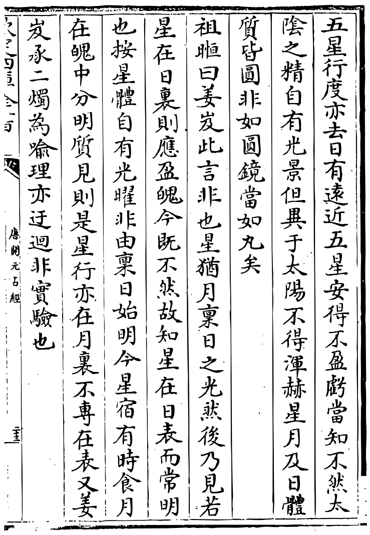


Figure 4. Zu Zuan’s discussion of a statement by Jiang Ji, *Kaiyuan zhanjing* 開元占經, *Siku quanshu* edn, 1: 23b. www.archive.org/details/06060533.cn.



五星行度亦去日有遠近五星安得不盈虧當知不然太
陰之精自有光景但異于太陽不得渾赫星月及日體
質皆圓非如圓鏡當如丸矣
祖暅曰姜岌此言非也星猶月稟日之光然後乃見若
星在日裏則應盈魄今既不然故知星在日表而常明
也按星體自有光曜非由稟日始明今星宿有時食月
在魄中分明質見則是星行亦在月裏不專在表又姜
岌承二燭為喻理亦迂迴非實驗也

唐開元占經

Figure 5. Dyke, Problem 3 in *Jigu suanjing*.

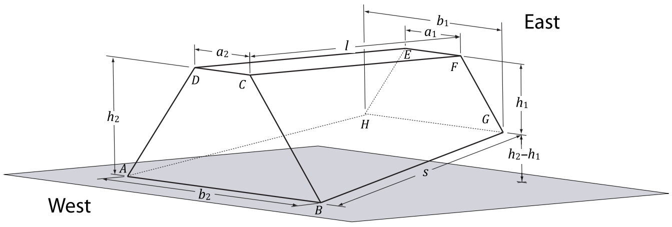


Figure 6. Volume dissection for Problem 3 of *Jigu suanjing*.

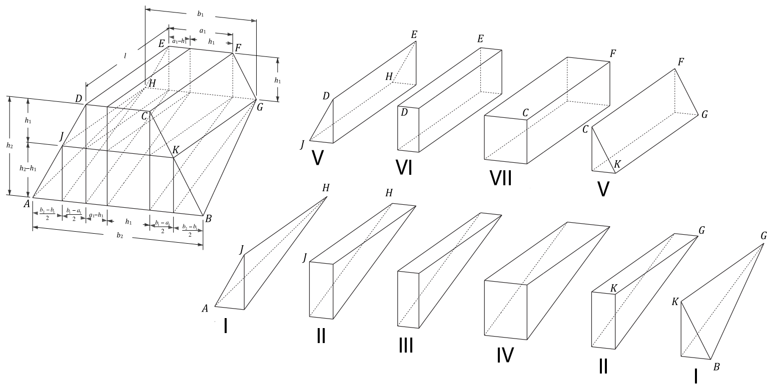
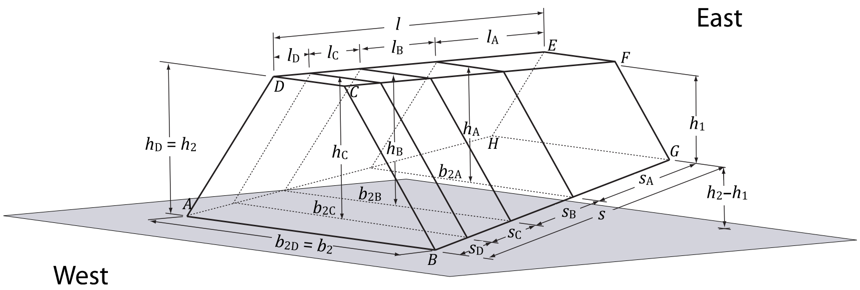
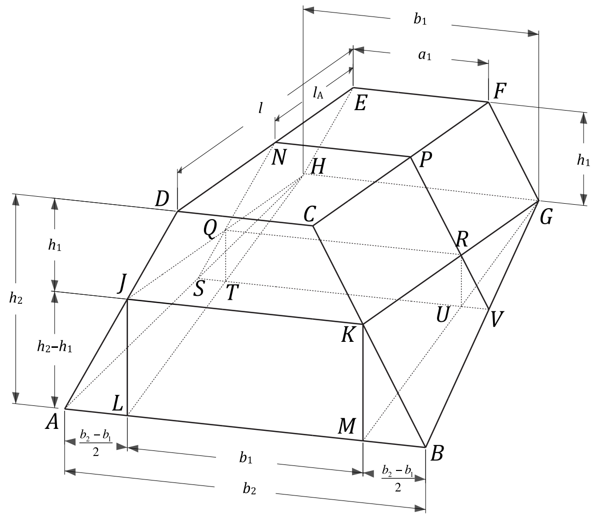


Figure 7. Shares of the counties in Problem 3 of *Jigu suanjing*.



Figures 8 and 9. Volume dissection for the calculation of county A’s share in Problem 3 of *Jigu suanjing*.



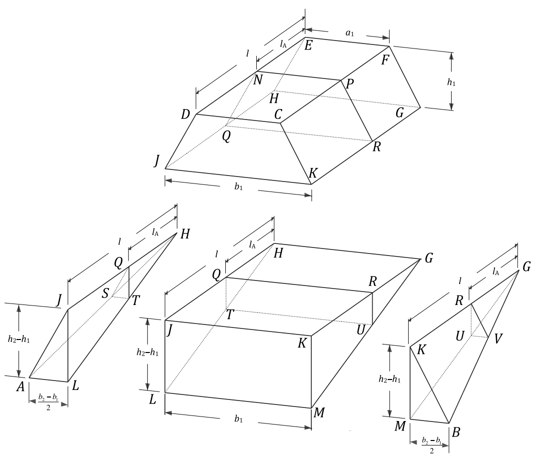


Figure 10. Canal, Problem 27 of *Hefang tongyi*.

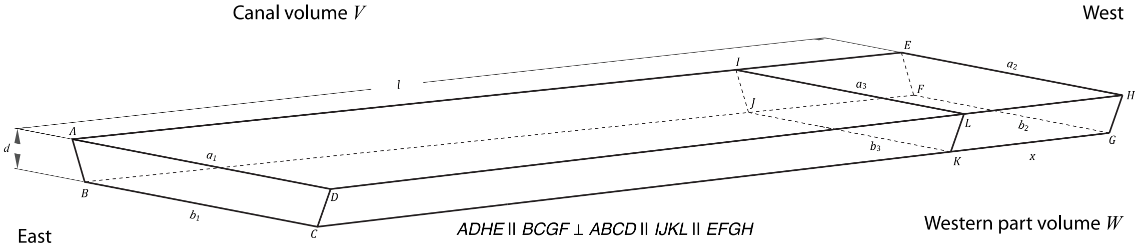


Figure Q. [No caption.]